Q-VECTOR DERIVATION

• QG GOVERNING EQUATIONS

$$\left.\begin{aligned}(1)\quad & \frac{D_g u_g}{Dt} = f_0 v_a + \beta y v_g \\ (2)\quad & \frac{D_g v_g}{Dt} = -f_0 u_a - \beta y u_g\end{aligned}\right\}\ \text{HORIZONTAL MOMENTUM}$$

$$(3)\quad \frac{D_g}{Dt}\left(-\frac{\partial \Phi}{\partial p}\right) = \sigma\omega \qquad \alpha = -\frac{\partial\Phi}{\partial p} = \frac{RT}{p} \qquad \frac{D_g T}{Dt} = \frac{\sigma p}{R}\omega \ \Big\}\ \text{THERMODYNAMIC ENERGY (ADIABATIC)}$$

$$\left.\begin{aligned}(4)\quad & f \approx \text{CONSTANT} \longrightarrow \nabla\cdot\vec{V}_g = 0 \longrightarrow \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \longrightarrow \frac{\partial u_g}{\partial x} = -\frac{\partial v_g}{\partial y} \\ & \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{\partial \omega}{\partial p} \qquad \begin{matrix} u = u_g + u_a \\ v = v_g + v_a \end{matrix} \longrightarrow \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} = -\frac{\partial\omega}{\partial p}\end{aligned}\right\}\ \text{CONTINUITY}$$

$$(5)\quad f_0\frac{\partial v_g}{\partial p} = -\frac{R}{p}\frac{\partial T}{\partial x} \qquad f_0\frac{\partial u_g}{\partial p} = \frac{R}{p}\frac{\partial T}{\partial y} \ \Big\}\ \text{THERMAL WIND BALANCE}$$

• SINCE $\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right)$, WE TAKE $f_0\frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt}\right)$ AND $\frac{R}{p}\frac{\partial}{\partial y}\left(\frac{D_g T}{Dt}\right)$

$$f_0\frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt}\right) = f_0\frac{\partial}{\partial p}\left(\frac{\partial u_g}{\partial t} + u_g\frac{\partial u_g}{\partial x} + v_g\frac{\partial u_g}{\partial y}\right) = f_0\frac{\partial}{\partial p}\left(f_0 v_a + \beta y v_g\right)$$

$$f_0\frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt}\right) = f_0\frac{\partial}{\partial t}\left(\frac{\partial u_g}{\partial p}\right) + f_0\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + f_0 u_g\frac{\partial}{\partial x}\left(\frac{\partial u_g}{\partial p}\right) + f_0\frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y} + f_0 v_g\frac{\partial}{\partial y}\left(\frac{\partial u_g}{\partial p}\right) = f_0^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p}$$

$$f_0\frac{\partial}{\partial p}\left(\frac{D_g u_g}{Dt}\right) = \frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) + f_0\left(\frac{\partial u_g}{\partial p}\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial p}\frac{\partial u_g}{\partial y}\right) = f_0^2\frac{\partial v_a}{\partial p} + f_0\beta y\frac{\partial v_g}{\partial p}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = -f_0\left(-\frac{R}{f_0 p}\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial y} - \frac{R}{f_0 p}\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial y}\right) + f_0^2\frac{\partial v_a}{\partial p} - f_0\beta y\frac{R}{f_0 p}\frac{\partial T}{\partial x}$$

*MAKE SUBSTITUTIONS BASED ON TW BALANCE AND CONTINUITY

$$\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial p}\right) = \underbrace{\frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)}_{-\sigma Q_2} + f_0^2\frac{\partial v_a}{\partial p} - \frac{R}{p}\beta y\frac{\partial T}{\partial x} \quad (6)$$

$$\frac{R}{p}\frac{\partial}{\partial y}\left(\frac{D_g T}{Dt}\right) = \frac{R}{p}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial t} + u_g\frac{\partial T}{\partial x} + v_g\frac{\partial T}{\partial y}\right) = \sigma\frac{\partial\omega}{\partial y}$$

$$\frac{R}{p}\frac{\partial}{\partial y}\left(\frac{D_g T}{Dt}\right) = \frac{R}{p}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial y}\right) + \frac{R}{p}\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{R}{p}u_g\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) + \frac{R}{p}\frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y} + \frac{R}{p}v_g\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) = \sigma\frac{\partial\omega}{\partial y}$$

$$\frac{R}{p}\frac{\partial}{\partial y}\left(\frac{D_g T}{Dt}\right) = \frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) + \frac{R}{p}\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right) = \sigma\frac{\partial\omega}{\partial y}$$

$$\frac{D_g}{Dt}\left(\frac{R}{p}\frac{\partial T}{\partial y}\right) = -\frac{R}{p}\underbrace{\left(\frac{\partial u_g}{\partial y}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial T}{\partial y}\right)}_{\sigma Q_2} + \sigma\frac{\partial\omega}{\partial y} \quad (7)$$

• SUBTRACT EQ (6) FROM EQ (7) AND RE-ARRANGE

$$\frac{D_g}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial y}\right) = \sigma Q_2 + \sigma\frac{\partial \omega}{\partial y} \qquad Q_2 = -\frac{R}{\sigma P}\frac{\partial \vec{V}_g}{\partial y}\cdot\vec{\nabla}_p T$$

$$-\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial P}\right) = -\sigma Q_2 + f_0^2\frac{\partial v_a}{\partial P} - \frac{R}{P}\beta y\frac{\partial T}{\partial x}$$

$$\sigma\frac{\partial \omega}{\partial y} - f_0^2\frac{\partial v_a}{\partial P} = -2\sigma Q_2 - \frac{R}{P}\beta y\frac{\partial T}{\partial x} \quad \text{OR} \quad \frac{\partial \omega}{\partial y} - \frac{f_0^2}{\sigma}\frac{\partial v_a}{\partial P} = -2Q_2 - \frac{R}{\sigma P}\beta y\frac{\partial T}{\partial x} \quad (8)$$

• SINCE $\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) = \frac{D_g}{Dt}\left(-\frac{R}{P}\frac{\partial T}{\partial x}\right)$, WE TAKE $f_0\frac{\partial}{\partial P}\left(\frac{D_g v_g}{Dt}\right)$ AND $\frac{R}{P}\frac{\partial}{\partial x}\left(\frac{D_g T}{Dt}\right)$

$$f_0\frac{\partial}{\partial P}\left(\frac{D_g v_g}{Dt}\right) = f_0\frac{\partial}{\partial P}\left(\frac{\partial v_g}{\partial t} + u_g\frac{\partial v_g}{\partial x} + v_g\frac{\partial v_g}{\partial y}\right) = f_0\frac{\partial}{\partial P}\left(-f_0 u_a - \beta y u_g\right)$$

$$f_0\frac{\partial}{\partial P}\left(\frac{D_g v_g}{Dt}\right) = \underline{f_0\frac{\partial}{\partial t}\left(\frac{\partial v_g}{\partial P}\right)} + f_0\frac{\partial u_g}{\partial P}\frac{\partial v_g}{\partial x} + \underline{f_0 u_g\frac{\partial}{\partial x}\left(\frac{\partial v_g}{\partial P}\right)} + f_0\frac{\partial v_g}{\partial P}\frac{\partial v_g}{\partial y} + \underline{f_0 v_g\frac{\partial}{\partial y}\left(\frac{\partial v_g}{\partial P}\right)} = -f_0^2\frac{\partial u_a}{\partial P} - f_0\beta y\frac{\partial u_g}{\partial P}$$

$$f_0\frac{\partial}{\partial P}\left(\frac{D_g v_g}{Dt}\right) = \frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) + f_0\left(\frac{\partial u_g}{\partial P}\frac{\partial v_g}{\partial x} + \frac{\partial v_g}{\partial P}\frac{\partial v_g}{\partial y}\right) = -f_0^2\frac{\partial u_a}{\partial P} - f_0\beta y\frac{\partial u_g}{\partial P}$$

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) = -f_0\left(\frac{R}{f_0 P}\frac{\partial T}{\partial y}\frac{\partial v_g}{\partial x} + \frac{R}{f_0 P}\frac{\partial T}{\partial x}\frac{\partial u_g}{\partial x}\right) - f_0^2\frac{\partial u_a}{\partial P} - f_0\beta y\frac{R}{f_0 P}\frac{\partial T}{\partial y}$$

* MAKE SUBSTITUTIONS BASED ON TW BALANCE AND CONTINUITY

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) = \underbrace{-\frac{R}{P}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)}_{\sigma Q_1} - f_0^2\frac{\partial u_a}{\partial P} - \frac{R}{P}\beta y\frac{\partial T}{\partial y} \quad (9)$$

$$\frac{R}{P}\frac{\partial}{\partial x}\left(\frac{D_g T}{Dt}\right) = \frac{R}{P}\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial t} + u_g\frac{\partial T}{\partial x} + v_g\frac{\partial T}{\partial y}\right) = \sigma\frac{\partial \omega}{\partial x}$$

$$\frac{R}{P}\frac{\partial}{\partial x}\left(\frac{D_g T}{Dt}\right) = \underline{\frac{R}{P}\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial x}\right)} + \frac{R}{P}\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \underline{\frac{R}{P}u_g\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)} + \frac{R}{P}\frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y} + \underline{\frac{R}{P}v_g\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right)} = \sigma\frac{\partial \omega}{\partial x}$$

$$\frac{R}{P}\frac{\partial}{\partial x}\left(\frac{D_g T}{Dt}\right) = \frac{D_g}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial x}\right) + \frac{R}{P}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right) = \sigma\frac{\partial \omega}{\partial x}$$

$$\frac{D}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial x}\right) = \underbrace{-\frac{R}{P}\left(\frac{\partial u_g}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial v_g}{\partial x}\frac{\partial T}{\partial y}\right)}_{\sigma Q_1} + \sigma\frac{\partial \omega}{\partial x} \quad (10)$$

• ADD EQ (9) AND EQ (10) AND REARRANGE

$$\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) = \sigma Q_1 - f_0^2\frac{\partial u_a}{\partial P} - \frac{R}{P}\beta y\frac{\partial T}{\partial y} \qquad Q_1 = -\frac{R}{\sigma P}\frac{\partial \vec{V}_g}{\partial x}\cdot\vec{\nabla}_p T$$

$$+\frac{D_g}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial x}\right) = \sigma Q_1 + \sigma\frac{\partial \omega}{\partial x}$$

$$\sigma\frac{\partial \omega}{\partial x} - f_0^2\frac{\partial u_a}{\partial P} = -2\sigma Q_1 + \frac{R}{P}\beta y\frac{\partial T}{\partial y} \quad \text{OR} \quad \frac{\partial \omega}{\partial x} - \frac{f_0^2}{\sigma}\frac{\partial u_a}{\partial P} = -2Q_1 + \frac{R}{\sigma P}\beta y\frac{\partial T}{\partial y} \quad (11)$$

- TAKE $\frac{\partial}{\partial x}\left(EQ(11)\right)$ AND $\frac{\partial}{\partial y}\left(EQ(8)\right)$

$$\frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial x}\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial x}\left(\frac{\partial u_a}{\partial P}\right) = -2\frac{\partial Q_1}{\partial x} + \frac{R}{\sigma P}\frac{\partial}{\partial x}\left(\beta y \frac{\partial T}{\partial y}\right)$$

$$\frac{\partial^2 \omega}{\partial x^2} - \frac{f_0^2}{\sigma}\frac{\partial}{\partial P}\left(\frac{\partial u_a}{\partial x}\right) = -2\frac{\partial Q_1}{\partial x} + \frac{R}{\sigma P}\beta y \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) \quad (12)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \omega}{\partial y}\right) - \frac{f_0^2}{\sigma}\frac{\partial}{\partial y}\left(\frac{\partial v_a}{\partial P}\right) = -2\frac{\partial Q_2}{\partial y} - \frac{R}{\sigma P}\frac{\partial}{\partial y}\left(\beta y \frac{\partial T}{\partial x}\right)$$

$$\frac{\partial^2 \omega}{\partial y^2} - \frac{f_0^2}{\sigma}\frac{\partial}{\partial P}\left(\frac{\partial v_a}{\partial y}\right) = -2\frac{\partial Q_2}{\partial y} - \frac{R}{\sigma P}\beta\frac{\partial T}{\partial x} - \frac{R}{\sigma P}\beta y \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial y}\right) \quad (13)$$

- ADD EQ (12) AND EQ (13)

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} - \frac{f_0^2}{\sigma}\frac{\partial}{\partial P}\left(\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}\right) = -2\frac{\partial Q_1}{\partial x} - 2\frac{\partial Q_2}{\partial y} - \frac{R}{\sigma P}\beta\frac{\partial T}{\partial x}$$

* SUBSTITUTE $-\frac{\partial \omega}{\partial P}$ FOR $\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y}$ (CONTINUITY)

$$\nabla_H^2 \omega + \frac{f_0^2}{\sigma}\frac{\partial^2 \omega}{\partial P^2} = -2\vec{\nabla}\cdot\vec{Q} - \frac{R}{\sigma P}\beta\frac{\partial T}{\partial x} \quad \text{OR} \quad \left(\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial P^2}\right)\omega = -2\vec{\nabla}\cdot\vec{Q} - \frac{R}{\sigma P}\beta\frac{\partial T}{\partial x}$$

- SINCE THE β-TERM IS GENERALLY SMALL IN THE MIDLATITUDES, WE CAN MAKE THE APPROXIMATION: $\left(\nabla_H^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial P^2}\right)\omega \approx -2\vec{\nabla}\cdot\vec{Q}$

$$\vec{Q} = -\frac{R}{\sigma P}\langle Q_1, Q_2\rangle = -\frac{R}{\sigma P}\left\langle \frac{\partial \vec{V}_g}{\partial x}\cdot\vec{\nabla}_P T,\ \frac{\partial \vec{V}_g}{\partial y}\cdot\vec{\nabla}_P T\right\rangle$$

- PHYSICAL INTERPRETATION OF Q-VECTORS

(6) $\frac{D_g}{Dt}\left(f_0\frac{\partial u_g}{\partial P}\right) = -\sigma Q_2 + f_0^2\frac{\partial v_a}{\partial P} - \frac{R}{P}\beta y\frac{\partial T}{\partial x}$

(7) $\frac{D_g}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial y}\right) = \sigma Q_2 + \sigma\frac{\partial \omega}{\partial y}$

(9) $\frac{D_g}{Dt}\left(f_0\frac{\partial v_g}{\partial P}\right) = \sigma Q_1 - f_0^2\frac{\partial u_a}{\partial P} - \frac{R}{P}\beta y\frac{\partial T}{\partial y}$

(10) $\frac{D_g}{Dt}\left(\frac{R}{P}\frac{\partial T}{\partial x}\right) = \sigma Q_1 + \sigma\frac{\partial \omega}{\partial x}$

* SINCE EQ(6) = EQ(7) AND EQ(9) = −EQ(10), WE CAN SEE THAT Q_1 AND Q_2 ACT TO DESTROY TW BALANCE

* AN AGEOSTROPHIC CIRCULATION ($u_a, v_a, \omega \neq 0$) IS REQUIRED TO RESTORE TW BALANCE