Cancellation Problem in QG Omega Eq.

Ref: Bluestein I (sect 5.6.4, 5.7.3)
Holton (sect 6.4.1)

QG Omega Eq:

$$\left(\nabla_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p (\zeta_g + f)\right]}_{A} \underbrace{- \frac{R}{\sigma p}\nabla_p^2 \left(-\vec{V}_g \cdot \nabla T\right)}_{B}$$

$$\left\{ \begin{aligned} \text{Alternate RHS} &= \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla \left(\frac{1}{f_o}\nabla^2 \Phi + f\right)\right] \\ &\quad + \frac{1}{\sigma}\nabla^2\left[\vec{V}_g \cdot \nabla \left(-\partial \Phi / \partial p\right)\right] \end{aligned} \right\}$$

Check for Galilean invariancy (terms don't change in different coordinate systems)

Add a constant velocity, $\vec{c}$, to the wind field everywhere. Effect is like hopping onto a reference frame that is moving at a velocity $-\vec{c}$ relative to the original reference frame.

Work on A above

$$-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-(\vec{V}_g + \vec{c}) \cdot \nabla_p (\zeta_g + f)\right] =$$

$$\underbrace{-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla_p (\zeta_g + f)\right]}_{C} \underbrace{- \frac{f_o}{\sigma}\left(-\vec{c} \cdot \nabla_p \frac{\partial \zeta_g}{\partial p}\right)}_{D}$$

$\therefore$ By changing the coordinate system term A has decreased by the amount given by term D:

$$-\frac{f_o}{\sigma}\left(-\vec{C}\cdot\nabla_p\frac{\partial\zeta_g}{\partial p}\right)$$

$\therefore$ Differential vorticity advection is not Galilean invariant, but vorticity <u>is</u> Galilean invariant.

Now work on the LaPlacian of thermal advection term B

$$-\frac{R}{\sigma p}\nabla_p^2\left[-(\vec{V}_g+\vec{C})\cdot\nabla_p T\right] = \overset{E}{-\frac{R}{\sigma p}\nabla_p^2\left(-\vec{V}_g\cdot\nabla T\right)}$$

$$\underset{F}{-\frac{R}{\sigma p}\nabla_p^2\left(-\vec{C}\cdot\nabla_p T\right)}$$

The extra term, F, can be modified through use of the thermal wind eq:

$$-\frac{\partial\vec{V}_g}{\partial p} = \frac{R}{fp}\hat{k}\times\nabla_p T$$

and remembering that:

$$\zeta_g = \hat{k}\cdot\nabla\times\vec{V}_g$$

to yield:

$$-\frac{R}{\sigma p}\nabla_p^2\left[-(\vec{V}_g+\vec{c})\cdot\nabla_p T\right] = -\frac{R}{\sigma p}\nabla_p^2\left(-\vec{V}_g\cdot\nabla T\right) \quad E$$

$$+\frac{f_0}{\sigma}\left(-\vec{c}\cdot\nabla_p\frac{\partial \zeta_g}{\partial p}\right) \quad G$$

Note that term G is equal in magnitude but opposite in sign to term D

Bottom line: By computing the two forcing functions on the RHS of the QG omega equation in a coordinate system moving with constant velocity, $\vec{c}$, we see that the "decrease" in vorticity advection forcing has been exactly offset by an increase in thermal advection forcing.

Further bottom line: The total forcing on the RHS of the QG omega equation (neglecting diabatic heating and friction) is Galilean invariant even though the two individual forcing functions are not Galilean invariant

Goal: Combine two forcing functions into one Galilean invariant term.

Ref: Winn-Nielsen (1959), Hoskins et al. (1978) and

Bluestein (section 5.7.3, pp 346-349) shows that that:

$$-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla_p\left(\zeta_g+f\right)\right]$$

can be written as two terms:

$$-\frac{f_o}{\sigma}\left(\underbrace{-\frac{\partial\vec{V}_g}{\partial p}\cdot\nabla_p\left[\zeta_g+f\right]}_{\substack{\text{advection of geostrophic}\\ \text{absolute vorticity by the}\\ \text{thermal wind}}}-\underbrace{\vec{V}_g\cdot\nabla_p\frac{\partial\zeta_g}{\partial p}}_{\substack{\text{advection of}\\ \text{thermal vorticity}\\ \text{by the geostrophic wind}}}\right)$$

Similarly, the temperature advection term can be written (with neglect of the cumbersome geostrophic deformation terms) as:

$$\frac{f_o}{\sigma}\left[\frac{\partial\vec{V}_g}{\partial p}\cdot\nabla_p\zeta_g+\left(-\vec{V}_g\cdot\nabla_p\frac{\partial\zeta_g}{\partial p}\right)\right]$$

When these two forcing functions are added the terms involving the advection of thermal vorticity by the geostrophic wind cancel

The final form of the QG omega equation with these simplifications becomes:

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\left[2\left(\frac{\partial \vec{V}_g}{\partial p}\cdot\nabla_p \xi_g\right) + \frac{\partial \vec{V}_g}{\partial p}\cdot\nabla_p f\right]$$

which to good approximation can be written as.

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{2f_0}{\sigma}\underbrace{\left(\frac{\partial \vec{V}_g}{\partial p}\cdot\nabla \xi_g\right)}_{\text{advection of geostrophic relative vorticity by the thermal wind}}$$

(This is eq. 5.7.42 in Bluestein (I))

Note: The corresponding equation in Holton (4th, 2004) is missing a factor of two (eq. 6.36)

Further note: The full form of Bluestein eq. 5.7.42 which includes the geostrophic deformation terms is given by eq. 5.7.40. A lot of algebraic manipulation is required to get the final form of the geostrophic deformation term in eq. 5.7.40. Although this neglected term is generally considerably smaller than the dominant advection of geostrophic relative vorticity by the thermal wind term, it can be locally significant near frontal regions.