2 Some Mathematical & Conceptual Tools: Part 1. Statistics

Turbulence is an intrinsic part of the atmospheric boundary layer that must be quantified in order to study it. The randomness of turbulence makes deterministic description difficult. Instead, we are forced to retreat to the use of statistics, where we are limited to average or expected measures of turbulence. In this chapter we review some basic statistical methods and show how measurements of turbulence can be put into a statistical framework. Usually, this involves separating the turbulent from the nonturbulent parts of the flow, followed by averaging to provide the statistical descriptor.

The role of spectra in this separation of parts is also described. Although at first glance we see a labyrinth of motions, turbulence may be idealized as consisting of a variety of different-size swirls or eddies. These eddies behave in a well-ordered manner when displayed in the form of a spectrum.

Statistical descriptors such as the variance or covariance are of limited usefulness unless we can physically interpret them. Variances are shown to be measures of turbulence intensity or turbulence kinetic energy, and covariances are shown to be measures of flux or stress. Flux and stress concepts are explored further, and Einstein's summation notation is introduced as a shorthand way to write these variables.

The concepts developed in this chapter are used extensively in the remainder of the book to help describe the turbulent boundary layer.

2.1 The Signature of Turbulence and Its Spectrum

Since we have lived most of our lives within the turbulent boundary layer, we have developed feelings or intuitions about the nature of turbulence that can be refined to help us classify and describe this phenomenon. Consider figure 2.1 for example, which shows the near-surface wind speed measured during a 2.5 hour period. A number of features stand out.

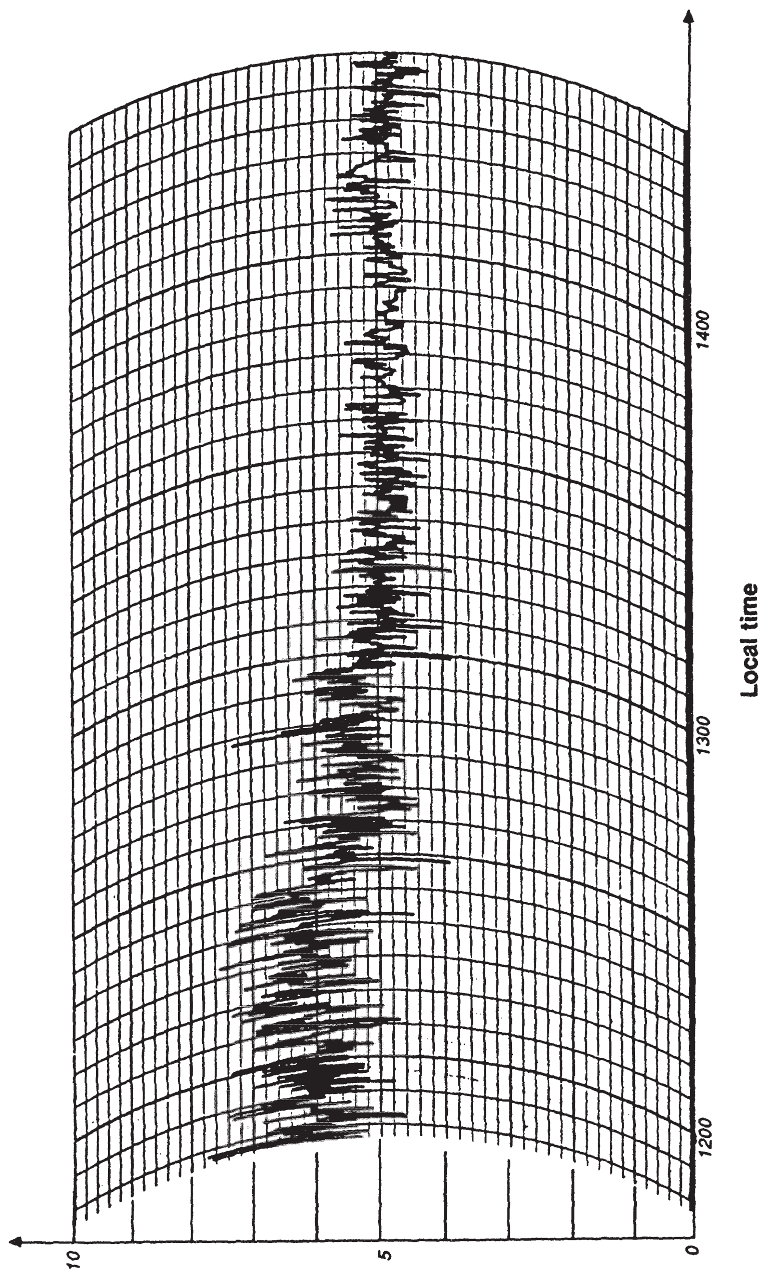


Fig. 2.1 Trace of wind speed obseved in early afternoon.

- The wind speed varies in an irregular pattern — a characteristic signature of turbulence. This quasi-randomness is what makes turbulence different from other motions, like waves.
- We can visually pick out a mean, or typical, value of the wind speed. For example, between noon and 1230 local time the average wind speed is about 6 m/s, while a bit later (between 1400 and 1430) the winds have decreased to about 5 m/s on the average. The ability to find a statistically-stable mean value suggests that turbulence is not completely random.
- The wind speeds do not vary from 0 to 100 m/s in this graph, but rather vary over a limited range of speeds. In other words, there is a measurable and definable intensity to the turbulence that shows up on this graph as the vertical spread of wind speed. The turbulence intensity appears to decrease between noon and 1400 local time.

  Near noon the instantaneous wind speed is often 1 m/s faster or slower than the mean, while at 1400 the wind speed varies by only about 0.5 m/s about its mean. Such a bounded characteristic of the wind speed means that we can use statistics such as the variance or standard deviation to characterize the turbulence intensity.
- There appears to be a wide variety of time-scales of wind variation superimposed on top of each other. If we look closely we see that the time period between each little peak in wind speed is about a minute. The larger peaks seem to happen about every 5 min. There are other variations that indicate a 10 min time period. The smallest detectable variations on this chart are about 10 s long.

  If each of these time variations is associated with a different size turbulent eddy (Taylor's hypothesis. See exercise 5 in Chapter 1), then we can conclude that we are seeing the signature of eddies ranging in size from about 50 m to about 3000 m. In other words, we are observing evidence of the spectrum of turbulence.

The turbulence spectrum is analogous to the spectrum of colors that appears when you shine a light through a prism. White light consists of many colors (i.e., many wavelengths or frequencies) superimposed on one another. The prism is a physical device that separates the colors. We could measure the intensity of each color to learn the magnitude of its contribution to the original light beam. We can perform a similar analysis on a turbulent signal using mathematical rather than physical devices to learn about the contribution of each different size eddy to the total turbulence kinetic energy.

Figure 2.2 shows an example of the spectrum of wind speed measured near the ground. The ordinate is a measure of the portion of turbulence energy that is associated with a particular size eddy. The abscissa gives the eddy size in terms of the time period and frequency of the wind-speed variation. Small eddies have shorter time periods than large eddies (again, using Taylor's hypothesis).

Peaks in the spectrum show which size eddies contribute the most to the turbulence kinetic energy. The leftmost peak with a period of near 100 h corresponds to wind speed variations associated with the passage of fronts and weather systems. In other words, there is evidence of the Rossby-wave cycle in our wind speed record. The next peak, at 24 h, shows the diurnal increase of wind speed during the day and decrease at night. The rightmost peak is the one we will study in this book. It indicates the microscale eddies having durations of 10 s to 10 min, just what we would have guessed from examining Fig 2.1 by eye.

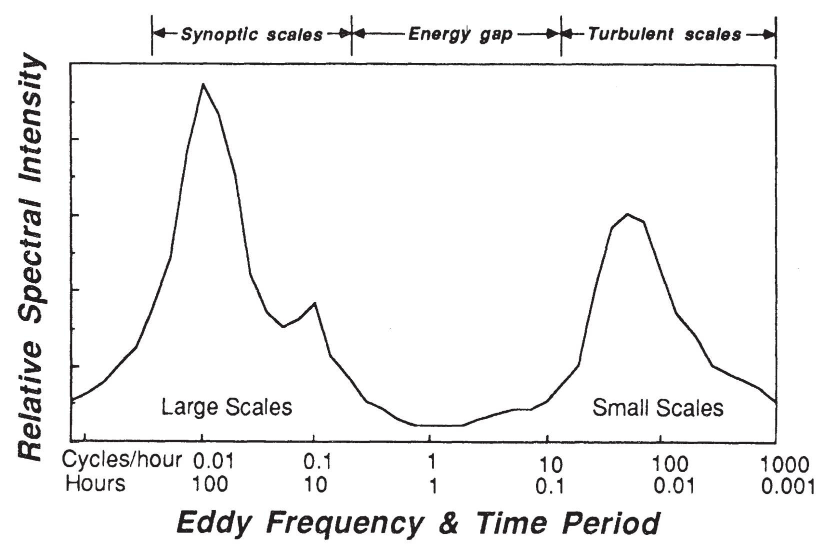


Fig. 2.2 Schematic spectrum of wind speed near the ground estimated from a study of Van der Hoven (1957).

Within the rightmost peak we see that the largest eddies are usually the most intense. The smaller, high frequency, eddies are very weak, as previously discussed. Large-eddy motions can create eddy-size wind-shear regions, which can generate smaller eddies. Such a net transfer of turbulence energy from the larger to the smaller eddies is known as the energy cascade. At the smallest size eddies, this cascade of energy is dissipated into heat by molecular viscosity. The flavor of this energy cascade was captured by Lewis Richardson in his 1922 poem:

Big whorls have little whorls,
Which feed on their velocity;
And little whorls have lesser whorls,
And so on to viscosity
(in the molecular sense).

In a later chapter we will describe the mathematical tools necessary to calculate spectra.

2.2 The Spectral Gap

There appears to be a distinct lack of wind-speed variation in Fig 2.1 having time periods of about 30 min to 1 h. The slow variation of the mean wind speed from 6 to 5 m/s over the 2 h period was already discussed. Shorter time periods were what we associated with microscale turbulence. The lack of variation at the intermediate time or space scales has been called the ***spectral gap***.

A separation of scales is evident in Fig 2.2, where the spectral gap appears as the large valley separating the microscale from the synoptic scale peaks. Motions to the left of the gap are said to be associated with the ***mean flow***. Motions to the right constitute ***turbulence***. The center of the gap is near the one-hour time period.

It is no accident that the response time used in Chapter 1 to define the BL is one hour. Implicit in the definition of the BL is the concept that turbulence is the primary agent for effecting changes in the BL. Hence, the spectral gap provides a means to separate the turbulent from the nonturbulent influences on the BL (in the microscale sense).

For some flows there might not be a spectral gap. For example, larger cumulus clouds act like large eddies with time scales on the order of an hour. Consequently, a spectrum of wind speed made in the cloud layer might not exhibit a vivid separation of scales. Most analyses of turbulence rely on the separation of scales to simplify the problem; hence, cloud-filled flow regimes might be difficult to properly describe.

Many of the operational numerical weather prediction models use grid spacings or wavelength cutoffs that fall within the spectral gap. This means that larger-scale motions can be explicitly resolved and deterministically forecast. The smaller-scale motions, namely turbulence, are not modeled directly. Rather, the effects of those subgrid scales on the larger scales are approximated. These smaller-size motions are said to be parameterized by subgrid-scale stochastic (statistical) approximations or models.

2.3 Mean and Turbulent Parts

There is a very easy way to isolate the large-scale variations from the turbulent ones. By averaging our wind speed measurements over a period of 30 minutes to one hour, we can eliminate or "average out" the positive and negative deviations of the turbulent velocities about the mean. Once we have the mean velocity, $\overline{U}$, for any time period, we can subtract it from the actual instantaneous velocity, U, to give us just the turbulent part, u':

$$u' = U - \overline{U} \tag{2.3a}$$

The existence of a spectral gap allows us to partition the flow field in this manner.

We can think of u' as the gust that is superimposed on the mean wind. It represents the part of the flow that varies with periods shorter than about one hour. The mean, $\overline{U}$, represents the part that varies with a period longer than about one hour.

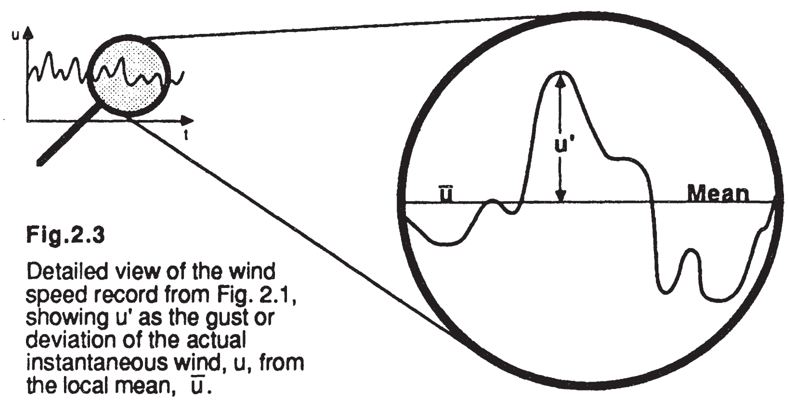


Fig.2.3

Detailed view of the wind speed record from Fig. 2.1, showing u' as the gust or deviation of the actual instantaneous wind, u, from the local mean, $\overline{u}$.

Fig 2.3 shows an expanded view of just a small portion of the wind trace from Fig 2.1. The straight line represents the mean wind over that portion of the record, while the wiggly line represents the actual instantaneous wind speed. The gust part, u', is sketched as the distance between those two lines. At some times the gust is positive, meaning the actual wind is faster than average. At other times the gust is negative, indicative of a slower than average wind.

Microscale turbulence is a three-dimensional phenomenon. Therefore, we expect that gusts in the x-direction might be accompanied by gusts in the y- and z-directions. Turbulence, by definition, is a type of motion. Yet motions frequently cause variations in the temperature, moisture, and pollutant fields if there is some mean gradient of that variable across the turbulent domain. Hence, we can partition each of the variables into mean and turbulent parts:

$$
\begin{aligned}
U &= \overline{U} + u' \\
V &= \overline{V} + v' \\
W &= \overline{W} + w' \qquad \qquad (2.3b) \\
\theta_v &= \overline{\theta_v} + \theta_v' \\
q &= \overline{q} + q' \\
c &= \overline{c} + c'
\end{aligned}
$$

Each of these terms varies in time and space (see Appendix B for a list of symbols).

2.4 Some Basic Statistical Methods

Because one of the primary avenues for studying turbulent flow is the stochastic approach, it is desirable to have a good working knowledge of statistics. This section will survey some of the basic methods of statistics, including the mean, variance, standard deviation, covariance, and correlation. Those readers having an adequate background on statistics might wish to skim this section.

2.4.1 The Mean

Time $\overline{(\ )}^{t}$, space $\overline{(\ )}^{s}$, and ensemble $\overline{(\ )}^{e}$ averages are three ways to define a mean. The ***time average*** applies at one specific point in space, and consists of a sum or integral over time period P. For any variable, A(t, s), that is a function of time, t, and space, s:

$$\overline{A}^{t}(s) = \frac{1}{N}\sum_{i=0}^{N-1} A(i, s) \qquad \text{or} \qquad \overline{A}^{t}(s) = \frac{1}{P}\int_{t=0}^{P} A(t, s)\, dt \tag{2.4.1a}$$

where $t = i\,\Delta t$ for the discrete case.

The ***spatial average***, which applies at some instant in time, is given by a sum or integral over spatial domain S:

$$\overline{A}^{s}(t) = \frac{1}{N}\sum_{j=0}^{N-1} A(t,j) \qquad \text{or} \qquad \overline{A}^{s}(t) = \frac{1}{S}\int_{t=0}^{S} A(t,s)\, ds \tag{2.4.1b}$$

where $s = j\Delta s$ in the discrete case.

An ***ensemble average*** consists of the sum over N identical experiments:

$$\overline{A}^{e}(t,s) = \frac{1}{N}\sum_{i=0}^{N-1} A_i(t,s) \tag{2.4.1c}$$

In the equations above, $\Delta t = P/N$ and $\Delta s = S/N$, where N is the number of data points.

For laboratory experiments, the ensemble average is the most desirable, because it allows us to reduce random experimental errors by repeating the basic experiment. Unlike laboratory experiments, however, we have little control over the atmosphere, so we are rarely able to observe reproducible weather events. We are therefore unable to use the ensemble average.

Spatial averages are possible by deploying an array of meteorological sensors covering a line, area, or volume. If the turbulence is ***homogeneous*** (statistically the same at every point in space, see Fig 2.4) then each of the sensors in the array will be measuring the same phenomenon, making a spatial average meaningful. The real atmosphere, however, is horizontally homogeneous in only limited locations, meaning that most spatial means are averaged over a variety of different phenomena. By proper choice of sensor-array domain size as well as intra-array spacing, one can sometimes isolate scales of phenomena for study, while averaging out the other scales.

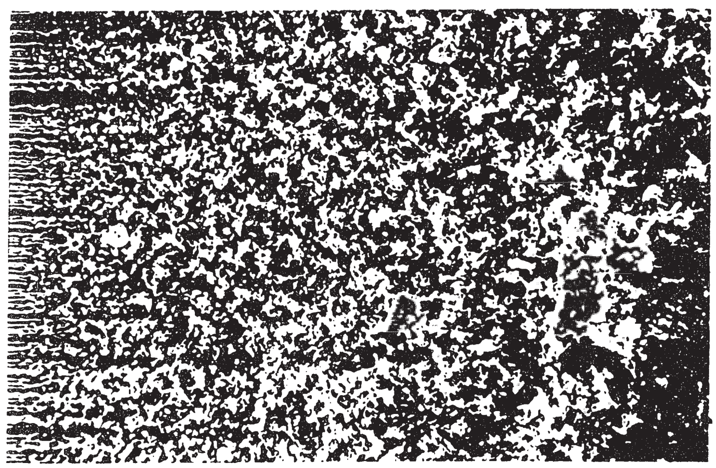


Fig. 2.4 Laboratory generation of homogeneous turbulence behind a grid. Using a finer grid than Fig 2.4, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation of homogeneous turbulence.(From Van Dyke, 1982)

Volume averaging is virtually impossible using direct sensors such as thermometers because of the difficulty of deploying these sensors at all locations and altitudes throughout the BL. Remote sensors such as radars, lidars, and sodars, however, can scan volumes of the atmosphere, making volume averages of selected variables possible. Details of these sensors are discussed in chapter 10.

Area averaging in the surface layer is frequently performed within small domains by deploying an array of small instrumented masts or instrument shelters on the ground. ***Line averages*** are similarly performed by erecting sensors along a road, for example.

Sensors mounted on a moving platform, such as a truck or an aircraft, can provide quasi-line averages. These are not true line (spatial) averages because the turbulence state of the flow may change during the time it takes the platform to move along the desired path. As a result, most measurement paths are designed as a compromise between long length (to increase the statistical significance by observing a larger number of data points) and short time (because of the diurnal changes that occur in the mean and turbulent state over most land surfaces).

Time averages are frequently used, and are computed from sensors mounted on a single, fixed-location platform such as a mast or tower. The relative ease of making observations at a fixed point has meant that time averaging has been the most popular in the lower BL. Some vertically-looking remote sensors also use this method to observe the middle and top of the BL. For turbulence that is both homogeneous and ***stationary*** (statistically not changing over time), the time, space and ensemble averages should all be equal. This is called the ***ergodic*** condition, which is often assumed to make the turbulence problem more tractable:

$$^{e}\overline{(\ )} = {}^{t}\overline{(\ )} = {}^{s}\overline{(\ )} \equiv \overline{(\ )} \tag{2.4.1d}$$

This book will use the overbar $\overline{(\ )}$ as an abbreviation for a generic average, not specifying whether it is a time, space, or ensemble average. Because of the popularity of the time average, however, many of our examples will use time as the independent variable.

2.4.2 Rules of Averaging

Let A and B be two variables that are dependent on time, and let c represent a constant. To find the average of the sum of A and B, we can employ the equations of the previous section with some basic rules of summation or integration to show that:

$$\overline{(A+B)} = \overline{A} + \overline{B} \tag{2.4.2a}$$

In terms of discrete sums, the average is:

$$\begin{aligned}
\overline{(A+B)} &= \frac{1}{N}\sum_{i=0}^{N-1}(A_i + B_i) \\
&= \frac{1}{N}\left(\sum_i A_i + \sum_i B_i\right) \\
&= \frac{1}{N}\sum_i A_i + \frac{1}{N}\sum_i B_i \\
&= \overline{A} + \overline{B}
\end{aligned}$$

In terms of continuous integrals:

$$\overline{(A + B)} = \frac{1}{P}\int_{t=0}^{P}(A + B)\,dt$$

$$= \frac{1}{P}\left(\int_t A\,dt + \int_t B\,dt\right)$$

$$= \frac{1}{P}\int_t A\,dt + \frac{1}{P}\int_t B\,dt$$

$$= \overline{A} + \overline{B}$$

Both the sum and integral approaches give the same answer, as expected.

We can use similar methods to show that:

$$\overline{(c\,A)} = c\,\overline{(A)} \tag{2.4.2b}$$

$$\overline{c} = c \tag{2.4.2c}$$

An important consequence of averaging is that an average value acts like a constant when averaged a second time over the same time period, P:

Define

$$\frac{1}{P}\int_{t=0}^{P} A(t,s)\,dt \equiv \overline{A}(P,s)$$

Therefore

$$\frac{1}{P}\int_{t=0}^{P}\overline{A}(P,s)\,dt \equiv \overline{A}(P,s)\,\frac{1}{P}\int_{t=0}^{P} dt$$

$$= \overline{A}(P,s)$$

Leaving

$$\overline{(\overline{A})} = \overline{A} \tag{2.4.2d}$$

Similarly, it can be shown that:

$$\overline{(\overline{A}\ B)} = \overline{A}\ \overline{B} \tag{2.4.2e}$$

Often we need to find the average of a derivative of a dependent variable. For example, let A be dependent on both t and s, where s is an independent variable such as x, y, or z. In this case, we must use ***Leibniz' theorem***:

$$\frac{d}{dt}\left[\int_{S_1(t)}^{S_2(t)} A(t,s)\, ds\right] = \int_{S_1(t)}^{S_2(t)} \left[\frac{\partial A(t,s)}{\partial t}\right] ds + A(t, S_2)\frac{d S_2}{dt} - A(t, S_1)\frac{d S_1}{dt} \tag{2.4.2f}$$

where S_1 and S_2 are the limits of integration.

For the special case where S_1 and S_2 are constant with time, we can simply interchange the order of integration and differentiation:

$$\frac{d}{dt}\left[\int_s A\, ds\right] = \int_s \left[\frac{\partial A}{\partial t}\right] ds$$

Multiplying both sides by 1/S gives, where $S = S_2 - S_1$:

$$\frac{d\left({}^{s}\overline{A}\right)}{dt} = {}^{s}\overline{\left(\frac{\partial A}{\partial t}\right)} \tag{2.4.2g}$$

This special case is not always valid for variable depth boundary layers.

Suppose we wish to find the time-rate-of-change of a BL-averaged mixing ratio, $\overline{r}$, where the BL average is defined by integrating over the depth of the BL; i.e., from z=0 to $z=z_i$. Since z_i varies with time, we can use the full Leibniz' theorem to give:

$$\frac{d}{dt}\left[z_i\ {}^{s}\overline{r}\right] = z_i\ {}^{s}\overline{\left[\frac{\partial r}{\partial t}\right]} + r(t, z_i^{+})\frac{d z_i}{dt} \tag{2.4.2h}$$

where z_i^{+} represents a location just above the top of the BL.

Finally, let's re-examine the spectral gap. If our averaging time is 30 minutes to 1 hour, turbulent fluctuations will be eliminated, leaving the longer-period time variations. As we saw in Fig 2.1, the 30-minute mean wind speed changes over the period of a few hours. Thus, we can take the 30-minute average of the time-derivative of variable A to find how $\overline{A}$ varies over longer periods:

$$\overline{\left(\frac{d\,A}{dt}\right)}^{t} = \frac{d\,\overline{A}^{t}}{dt} \tag{2.4.2i}$$

In other words, the average of the local slopes (slope = rate of change with time) equals the slope of the averages (see Fig 2.5).

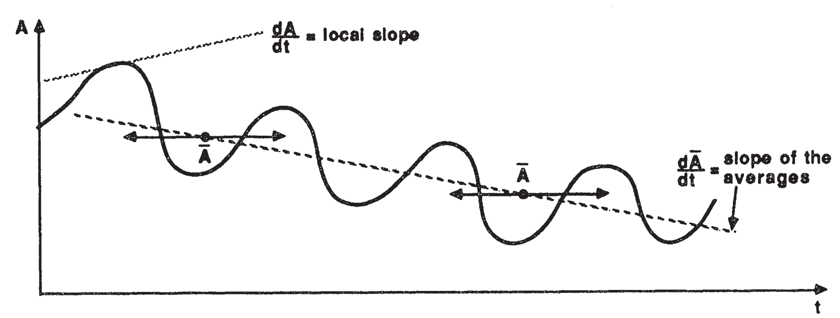


Fig. 2.5 Schematic comparison of the slope of average values verses the average of local slopes. Averaging period, P, is indicated by the width of the horizontal arrows.

This is a difficult concept that deserves some thought on the part of the reader. It is an important consequence of the spectral gap because it allows us to make a deterministic forecast of a mean variable such as $\overline{A}$ using simplified, stochastic, representations of the turbulence. Otherwise, operational forecasts of seemingly simple variables such as temperature or wind would be much more difficult.

To summarize the rules of averaging:

$$\begin{aligned}
\overline{c} &= c \\
\overline{(c\,A)} &= c\,\overline{A} \\
\overline{(\overline{A})} &= \overline{A} \\
\overline{(\overline{A}\,B)} &= \overline{A}\,\overline{B} \\
\overline{(A+B)} &= \overline{A} + \overline{B} \\
\overline{\left(\frac{d\,A}{dt}\right)} &= \frac{d\,\overline{A}}{dt}
\end{aligned} \tag{2.4.2k}$$

2.4.3 Reynolds Averaging

The averaging rules of the last section can now be applied to variables that are split into mean and turbulent parts. Let $A = \overline{A} + a'$ and $B = \overline{B} + b'$. Starting with the instantaneous value, A, for example, we can find its mean using the fifth and third rules of the previous section:

$$\overline{(A)} = \overline{(\overline{A} + a')} = \overline{(\overline{A})} + \overline{a'} = \overline{A} + \overline{a'}$$

The only way that the left and right sides can be equal is if

$$\overline{a'} = 0 \tag{2.4.3a}$$

This result is not surprising if one remembers the definition of a mean value. By definition, the sum of the positive deviations from the mean must equal the sum of the negative deviations. Thus the deviations balance when summed, as implied in the above average.

Another example: start with the product $\overline{B}\,a'$ and find its average. Employing the above result together with the fourth averaging rule, we find that

$$\overline{(\overline{B}\,a')} = \overline{B}\,\overline{a'} = \overline{B} \cdot 0 = 0 \tag{2.4.3b}$$

Similarly, $\overline{\overline{A}\,b'} = 0$. One should not become too lax about the average of primed variables, as is demonstrated next.

The average of the product of A and B is

$$\begin{aligned}
\overline{(A \cdot B)} &= \overline{(\overline{A} + a')\,(\overline{B} + b')} \\
&= \overline{(\overline{A}\,\overline{B} + a'\,\overline{B} + \overline{A}\,b' + a'\,b')} \\
&= \overline{(\overline{A}\,\overline{B})} + \overline{(a'\,\overline{B})} + \overline{(\overline{A}\,b')} + \overline{(a'\,b')} \\
&= \overline{A}\,\overline{B} + 0 + 0 + \overline{a'\,b'} \\
&= \overline{A}\,\overline{B} + \overline{a'\,b'}
\end{aligned} \tag{2.4.3c}$$

The nonlinear product $\overline{a'\,b'}$ is NOT necessarily zero. The same conclusion holds for other nonlinear variables such as:

$$\overline{a'^2} \ , \ \overline{a'b'^2} \ , \ \overline{a'^2 b'^2} \ .$$

In fact, these nonlinear terms must be retained to properly model turbulence. This is a dramatic difference from many linear theories of waves, where the nonlinear terms are often neglected as a first-order approximation.

2.4.4 Variance, Standard Deviation and Turbulence Intensity

One statistical measure of the dispersion of data about the mean is the variance, σ^2, defined by:

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} (A_i - \overline{A})^2 \tag{2.4.4a}$$

This is known as the ***biased variance***. It is a good measure of the dispersion of a sample of BL observations, but not the best measure of the dispersion of the whole population of possible observations. A better estimate of the variance (an ***unbiased variance***) of the population, given a sample of data, is

$$\sigma_A^2 = \frac{1}{(N-1)} \sum_{i=0}^{N-1} (A_i - \overline{A})^2 \tag{2.4.4b}$$

When N is large, as it often is for turbulence measurements, $1/N \cong 1/(N-1)$. As a result, the biased definition is usually used in BL meteorology for convenience.

Recall that the turbulent part (or the perturbation or gust part) of a turbulent variable is given by $a' = A - \overline{A}$. Substituting this into the biased definition of variance gives

$$\sigma_A^2 = \frac{1}{N} \sum_{i=0}^{N-1} a'^{\,2}_i = \overline{a'^2} \tag{2.4.4c}$$

Thus, whenever we encounter the average of the square of a turbulent part of a variable, such as $\overline{u'^2}$, $\overline{v'^2}$, $\overline{w'^2}$, $\overline{\theta'^2}$, $\overline{r'^2}$, or $\overline{q'^2}$,we can interpret these as variances.

The standard deviation is defined as the square root of the variance:

$$\sigma_A = \left(\overline{a'^2}\right)^{1/2} \tag{2.4.4d}$$

The standard deviation always has the same dimensions as the original variable. Fig 2.6 shows the relationship between a turbulent trace of wind speed and the corresponding standard deviation. It can be interpreted as a measure of the magnitude of the spread or dispersion of the original data from its mean. For this reason, it is used as a measure of the intensity of turbulence. In figure 2.1, for example, we might guess the standard deviation to be about 0.5 - 0.6 m/s at noon, dropping to about 0.3 m/s by 1400 local time.

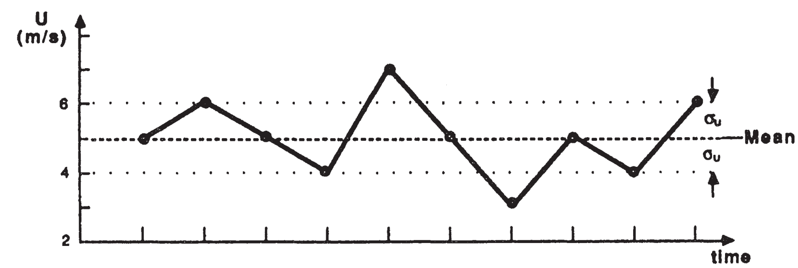


Fig. 2.6 Relationship between the standard deviation and turbulence variations. Solid line connects the data points; heavy dashed line is the average; dotted lines are drawn one standard deviation above and below the mean. Based on the example data in section 2.4.5.

Near the ground, the turbulence intensity might be expected to increase as the mean wind speed, M, increases. For this reason a dimensionless measure of the ***turbulence intensity***, I, is often defined as

$$I = \sigma_M / \overline{M} \tag{2.4.4e}$$

For mechanically generated turbulence, one might expect σ_M to be a simple function of M. As we learned in Chapter 1, $I < 0.5$ is required for Taylor's hypothesis to be valid.

2.4.5 Covariance and Correlation

In statistics, the covariance between two variables is defined as

$$\text{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} (A_i - \overline{A}) \cdot (B_i - \overline{B}) \tag{2.4.5a}$$

Using our Reynolds averaging methods, we can show that:

$$\text{covar}(A,B) \equiv \frac{1}{N} \sum_{i=0}^{N-1} a_i' \, b_i'$$

$$= \overline{a' \, b'} \qquad (2.4.5b)$$

Thus, the nonlinear turbulence products that were introduced in section 2.4.3 have the same meaning as covariances.

The covariance indicates the degree of common relationship between the two variables, A and B. For example, let A represent air temperature, T, and let B be the vertical velocity, w. On a hot summer day over land, we might expect the warmer than average air to rise (positive T' and positive w'), and the cooler than average air to sink (negative T' and negative w'). Thus, the product w'T' will be positive on the average, indicating that w and T vary together. The covariance $\overline{w'T'}$ is indeed found to be positive throughout the bottom 80% of the convective mixed layer.

Sometimes, one is interested in a normalized covariance. Such a relationship is defined as the ***linear correlation coefficient***, r_{AB}:

$$r_{AB} \equiv \frac{\overline{a' \, b'}}{\sigma_A \, \sigma_B} \qquad (2.4.5c)$$

This variable ranges between -1 and +1 by definition. Two variables that are perfectly correlated (i.e., vary together) yield r = 1. Two variables that are perfectly negatively correlated (i.e., vary oppositely) yield r = -1. Variables with no net variation together yield r = 0. Fig 2.7 shows typical correlation coefficients in the ML.

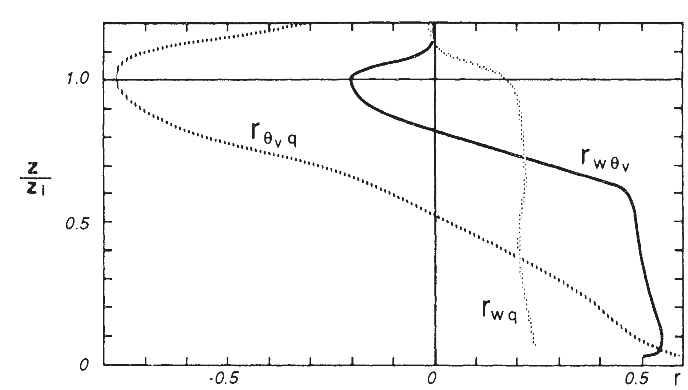


Fig. 2.7 Correlation coefficient profiles in the convective mixed layer.

2.4.6 Example

Problem. Suppose that we erect a short mast instrumented with anemometers to measure the U and W wind components. We record the instantaneous wind speeds every 6 s for a minute, resulting in the following 10 pairs of wind observations:

U (m/s):	5	6	5	4	7	5	3	5	4	6
W (m/s):	0	-1	1	0	-2	1	2	-1	1	-1

Find the mean, biased variance, and standard deviation for each wind component. Also, find the covariance and correlation coefficient between U and W.

Solutions.

$\overline{U} = 5\ \mathrm{m \cdot s^{-1}}$ $\quad \sigma_U^2 = 1.20\ \mathrm{m^2 \cdot s^{-2}}$ $\quad \sigma_U = 1.10\ \mathrm{m \cdot s^{-1}}$

$\overline{W} = 0\ \mathrm{m \cdot s^{-1}}$ $\quad \sigma_W^2 = 1.40\ \mathrm{m^2 \cdot s^{-2}}$ $\quad \sigma_W = 1.18\ \mathrm{m \cdot s^{-1}}$

$\overline{u'w'} = -1.10\ \mathrm{m^2 \cdot s^{-2}}$ $\quad r_{UW} = -0.85$ (dimensionless)

Discussion. Thus, the turbulent variations of W are more intense than those of U, even though the mean wind speed for W is zero in this example. U and W tend to vary in opposite directions on the average, as indicated by the negative values for the covariance and the correlation coefficient. The magnitude of the correlation coefficient is fairly high (close to one), meaning that there are just a few observations where U and W vary in the same direction, but many more observations where they vary oppositely.

2.5 Turbulence Kinetic Energy

The usual definition of kinetic energy (KE) is $KE = 0.5\ m\ M^2$, where m is mass. When dealing with a fluid such as air it is more convenient to talk about kinetic energy per unit mass, which is just $0.5\ M^2$.

It is enticing to partition the kinetic energy of the flow into a portion associated with the mean wind (MKE), and a portion associated with the turbulence (TKE). By taking advantage of the mean and turbulent parts of velocity introduced in section 2.3, we can immediately write the desired equations:

$$\mathrm{MKE}/m = \frac{1}{2}\left(\overline{U}^2 + \overline{V}^2 + \overline{W}^2\right) \tag{2.5a}$$

$$e = \frac{1}{2}\left(u'^2 + v'^2 + w'^2\right) \tag{2.5b}$$

where e represents an instantaneous turbulence kinetic energy per unit mass. There is an additional portion of the total KE consisting of mean-turbulence products, but this disappears upon averaging.

Rapid variations in the value of e with time can be expected as we measure faster and slower gusts. By averaging over these instantaneous values, we can define a mean turbulence kinetic energy (TKE) that is more representative of the overall flow:

$$\frac{\mathrm{TKE}}{m} = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right) = \bar{e} \tag{2.5c}$$

We can immediately see the relationship between TKE/m and the definition of variance defined in the last section. It is apparent that statistics will play an important role in our quantification of turbulence.

The turbulence kinetic energy is one of the most important quantities used to study the turbulent BL. We have already discussed in Chapter 1 that turbulence can be generated by buoyant thermals and by mechanical eddies. It is suppressed by statically stable lapse rates and dissipated into heat by the effects of molecular viscosity. By writing a budget equation for TKE, we can balance the production terms against the loss terms to determine whether the BL will become more turbulent, or whether turbulence will decay in the BL. This will be done in Chapter 5.

A typical daytime variation of TKE in convective conditions is shown in Fig 2.8. Examples of the vertical profile of TKE for various boundary layers are shown in Fig 2.9. During the daytime, buoyancy allows air parcels to accelerate in the middle of the ML, allowing $\overline{w'^2}$ to be large there and contributing to the total TKE (Fig 2.9a).

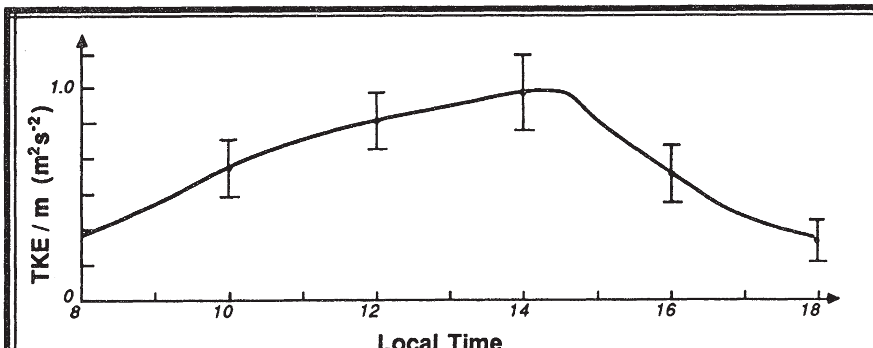


Fig. 2.8 Diurnal cycle of TKE/m = 1/2 ($\overline{u'^2}$+ $\overline{v'^2}$ +$\overline{w'^2}$) measured by aircraft below 300 m agl over Tennessee, August 1978. Adapted from Bean and Repoff, 1983.

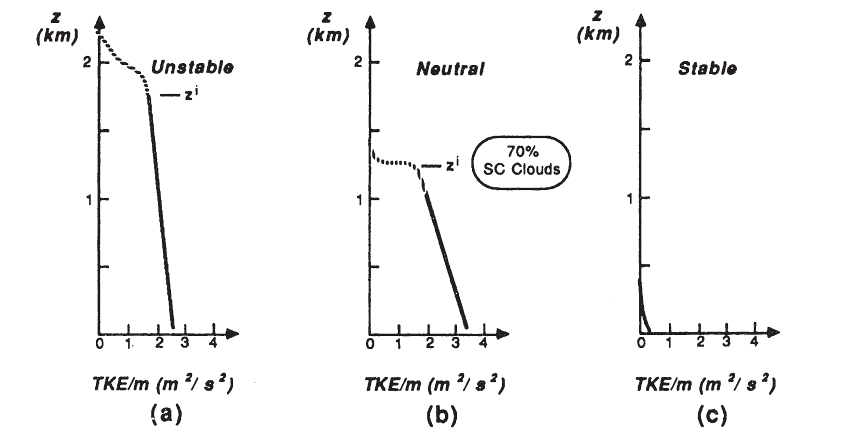


Fig. 2.9 Examples of turbulence kinetic energy per unit mass (TKE/m = 1/2 ($\overline{u'^2}+\overline{v'^2}+\overline{w'^2}$)) for various boundary layers: (a) Daytime convective mixed layer with mostly clear skies and light winds 1600 local time, 14 June 1983, Chickasha, OK; (b) Near neutral day with strong winds (10-15 m/s at surface) and broken clouds at 1100 local time, 12 June 1983, Chickasha, OK; (c) Nocturnal stable boundary layer at 1000 local time, 11 September 1973 in Minnesota (from Caughey, et al, 1979,) Part (c) data excludes $\overline{v'^2}$ contributions because of measurement errors.

On overcast days when there is little heating of the ground, wind shears and flow over obstacles create turbulence near the ground that gradually decreases intensity with height (Fig 2.9b). This turbulence is produced in primarily the $\overline{u'^2}$ and $\overline{v'^2}$ components. Days of both strong winds and strong heating will have both sources of turbulence.

For night, Fig 2.9c shows how the static stability suppresses the TKE, causing it to decrease rapidly with height. Turbulence is produced primarily near the ground by wind shears, although the enhanced shears near the nocturnal jet can also generate turbulence. Not apparent in this figure is the observation that nocturnal turbulence is sometimes sporadic: happening in turbulent bursts followed by quiescent periods.

2.6 Kinematic Flux

2.6.1 Definitions

Flux is the transfer of a quantity per unit area per unit time. In BL meteorology, we are often concerned with mass, heat, moisture, momentum and pollutant fluxes. The dimensions of these fluxes are summarized below, using SI units as the example:

Flux	Symbol	Units
mass	$\tilde{M}$	$\left[\frac{kg_{air}}{m^2 \cdot s}\right]$
heat	$\tilde{Q}_H$	$\left[\frac{J}{m^2 \cdot s}\right]$
moisture	$\tilde{R}$	$\left[\frac{kg_{water}}{m^2 \cdot s}\right]$
momentum	$\tilde{F}$	$\left[\frac{kg \cdot (m \cdot s^{-1})}{m^2 \cdot s}\right]$
pollutant	$\tilde{\chi}$	$\left[\frac{kg_{pollutant}}{m^2 \cdot s}\right]$ or $\left[\frac{kg_{pollutant}}{m^3} \cdot \frac{m}{s}\right]$

Sometimes the moisture flux is rewritten as a latent heat flux, $\tilde{Q}_E$, where $\tilde{Q}_E = L_v \tilde{R}$ and L_v is the latent heat of vaporization of water ($L_v \cong 2.45 \times 10^6$ J/kg at a summertime BL temperature of 20 °C).

As a reminder, momentum is mass times velocity (kg·m/s); thus, a momentum flux is (kg·m/s)/(m^2·s). These units are identical to N/m^2, which are the units for stress. The nature of stress is reviewed in section 2.9.

Unfortunately, we rarely measure quantities such as heat or momentum directly. Instead we measure things like temperature or wind speed. Therefore, for convenience the above fluxes can be redefined in ***kinematic form*** by dividing by the density of moist air, ρ_{air}. In the case of sensible heat flux, we also divide by the specific heat of air. In fact, the term $\rho\ C_p = 1.216 \times 10^3$ (W/m^2) / (K·m/s) allows us to easily convert between kinematic heat fluxes and normal heat fluxes.

Kinematic Flux	Symbol	Equation	Units	
mass		$M = \frac{\tilde{M}}{\rho_{air}}$	$\left[\frac{m}{s}\right]$	(2.6.1a)
heat		$Q_H = \frac{\tilde{Q}_H}{\rho_{air}\ C_{p_{air}}}$	$\left[K \frac{m}{s}\right]$	(2.6.1b)

$$\text{moisture} \qquad R = \frac{\tilde{R}}{\rho_{air}} \qquad \left[\frac{kg_{water}}{kg_{air}} \cdot \frac{m}{s}\right] \qquad (2.6.1c)$$

$$\text{momentum} \qquad F = \frac{\tilde{F}}{\rho_{air}} \qquad \left[\frac{m}{s} \cdot \frac{m}{s}\right] \qquad (2.6.1d)$$

$$\text{pollutant} \qquad \chi = \frac{\tilde{\chi}}{\rho_{air}} \qquad \left[\frac{kg_{pollutant}}{kg_{air}} \cdot \frac{m}{s}\right] \qquad (2.6.1e)$$

The above definitions are viable because the boundary layer is usually so thin that the density change across it can be neglected in comparison to changes of the other meteorological variables. For example, the standard atmospheric air density is 1.225 kg/m^3 at sea level and 1.112 kg/m^3 at 1000m, a difference of only 10%.

These kinematic fluxes are now expressed in units that we can measure directly: wind speed for mass and momentum fluxes; temperature and wind speed for heat flux; and specific humidity (q) and wind speed for moisture flux. The pollutant flux is frequently expressed in either form: concentration and wind speed, or mass ratio (like parts per million, ppm) and wind speed.

Each of these fluxes can be split into three components. For example, there might be a vertical component of heat flux, and two horizontal components of heat flux, as sketched in Fig 2.10. Similar fluxes could be expected for mass, moisture, and pollutants. Hence, we can picture these fluxes as vectors.

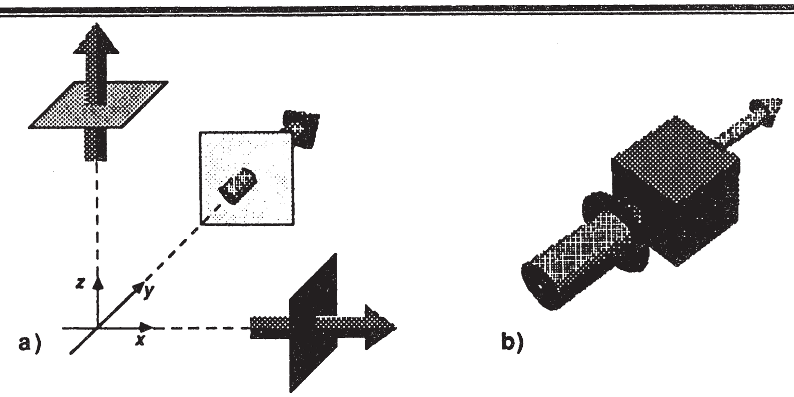
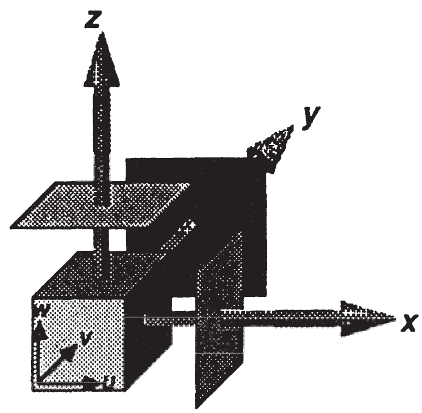


Fig. 2.10 Flux is the rate of transfer of a quantity across a unit area. (a) The net flux can be split into components in the x,y, and z directions, as shown here. (b) If a greater flux enters a volume than leaves, then there must be a net increase of that quantity within the volume.

For momentum, we have the added dimension that the flux in any one direction might be the flux of U, V or W momentum (see Fig 2.11). This means that there are nine components of this flux to consider: each of the three momentum components can pass through a plane normal to any of the three cartesian directions. The momentum flux is thus said to be a ***tensor***. Just for the record, this kind of tensor is known as a second order tensor. A vector is a first order tensor, and a scalar is a zero order tensor.

Fig. 2.11
Momentum can be split into the three cartesion directions, based on the u, v, and w components of wind. Momentum flux can consist of the transfer of any of these three components in any of three directions: x, y, and z, yielding a total of nine momentum flux components.



As you might guess, we can split the fluxes into mean and turbulent parts. For the flux associated with the mean wind (i.e., advection), it is easy to show, for example, that

Vertical kinematic advective heat flux $= \overline{W}\cdot\overline{\theta}$ (2.6.1f)

Vertical kinematic advective moisture flux $= \overline{W}\cdot\overline{q}$ (2.6.1g)

x-direction kinematic advective heat flux $= \overline{U}\cdot\overline{\theta}$ (2.6.1h)

Vertical kinematic advective flux of U-momentum $= \overline{W}\cdot\overline{U}$ (2.6.1i)

The last flux is also the kinematic flux of W-momentum in the x-direction.

Fluxes in other directions can be constructed in an analogous fashion. These fluxes have the proper dimensions for kinematic fluxes. They also make physical sense. For example, a greater vertical velocity or a greater potential temperature both create a greater vertical heat flux, as would be intuitively expected.

2.6.2 Example

Problem. Given $\tilde{Q}_H = 365\ \mathrm{W{\cdot}m^{-2}}$. Find Q_H.

Solution. $Q_H = \tilde{Q}_H / (\rho\, C_p)$

$$= (365\ \mathrm{Wm^{-2}}) / [(1.21\ \mathrm{kg{\cdot}m^{-3}}){\cdot}(1005\ \mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}})] = 0.30\ \mathrm{K{\cdot}m/s}$$

Discussion. This is a typical daytime kinematic heat flux during strong convection.

2.7 Eddy Flux

We saw in the last section that fluid motion can transport quantities, resulting in fluxes. Turbulence also involves motion. Thus we expect that turbulence transports quantities too.

2.7.1 Concepts

A term like $\overline{w'\theta'}$ looks similar to the kinematic flux terms of the last section, except that the perturbation values are used instead of the mean values of W and θ. If turbulence is completely random, then a positive $w'\theta'$ one instant might cancel a negative $w'\theta'$ at some later instant, resulting in a near zero value for the average turbulent heat flux. As is shown below, however, there are situations where the average turbulent flux might be significantly different from zero.

As a conceptual tool, suppose we examine a small idealized eddy near the ground on a hot summer day (see Fig 2.12a). The average potential temperature profile is usually superadiabatic in such surface layers. If the eddy is a swirling motion, then some of the air from position 1 will be mixed downward (i.e., w' is negative), while some air from position 2 will mix up (i.e., w' is positive) to take its place. The average motion caused by turbulence is $\overline{w'} = 0$, as expected (from section 2.4.3).

The downward moving air parcel (negative w') ends up being cooler than its surroundings (negative θ', assuming that θ' was conserved during its travel), resulting in an instantaneous product $w'\theta'$ that is positive. The upward moving air (positive w') is warmer than its surroundings (positive θ'), also resulting in a positive instantaneous product $w'\theta'$. Both the upward and downward moving air contribute positively to the flux, $w'\theta'$; thus, the average kinematic eddy heat flux $\overline{w'\theta'}$ is positive for this small-eddy mixing process.

This important result shows that turbulence can cause a net transport of a quantity such as heat ($\overline{w'\theta'} \neq 0$), even though there is no net transport of mass ($\overline{w'} = 0$). Turbulent eddies transport heat upward in this case, tending to make the lapse rate more adiabatic.

Next, let's examine what happens on a night where a statically stable lapse rate is present (Fig 2.12b). Again, picture a small eddy moving some air up and some back down. An upward moving parcel ends up cooler than its surrounding (negative $w'\theta'$), while a downward moving parcel is warmer (negative $w'\theta'$). The net effect of the small eddy is to cause a negative $\overline{w'\theta'}$, meaning a downward transport of heat.

Fruit growers utilize this process on cold nights as one method to prevent their fruit from freezing. They run motor-driven fans throughout the orchard to generate turbulent eddies. These eddies mix the warmer air down towards the fruit, and mix the cooler near-surface air upward out of the orchard, thereby potentially saving the crop.

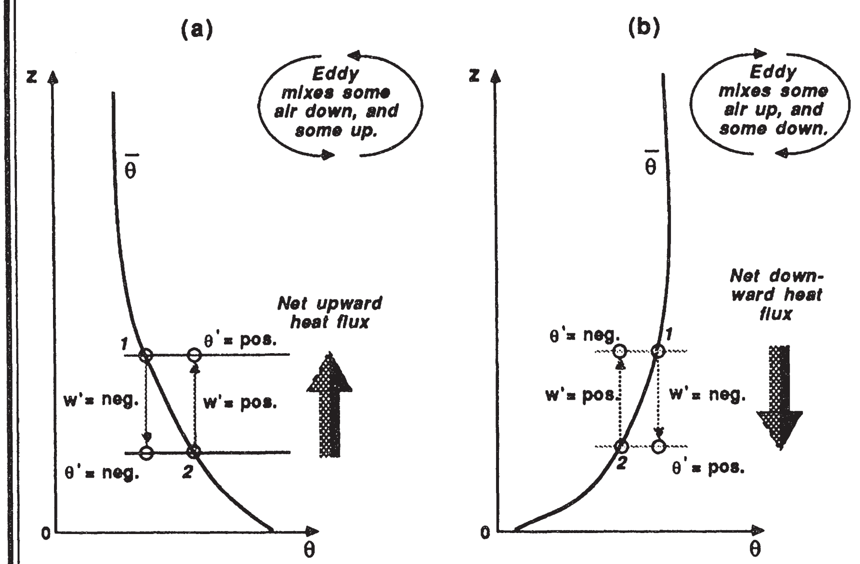


Fig. 2.12 Idealization of the small eddy mixing process, showing (a) net upward turbulent heat flux in a statically unstable environment, and (b) net downward turbulent heat flux in a stable environment.

Again we see the statistical nature of our description of turbulence. A kinematic flux such as $\overline{w'\theta'}$ is nothing more than a statistical covariance. We will usually leave out the word "kinematic" in future references to such fluxes.

As before, we can extend our arguments to write various kinds of eddy flux:

Vertical kinematic eddy heat flux $= \overline{w'\theta'}$ (2.7.1a)

Vertical kinematic eddy moisture flux $= \overline{w'q'}$ (2.7.1b)

x-direction kinematic eddy heat flux $= \overline{u'\theta'}$ (2.7.1c)

Vertical kinematic eddy flux of U-momentum $= \overline{u'w'}$ (2.7.1d)

The last flux is also the x-direction kinematic eddy flux of W-momentum.

Comparing the advective fluxes to the eddy fluxes, it is important to recognize that $\overline{W} \cong 0$ throughout most of the boundary layer. As a result, the vertical advective fluxes are usually negligible compared to the vertical turbulent fluxes. No such statement can be made about the horizontal fluxes, where strong mean horizontal winds and strong turbulence can cause fluxes of comparable magnitudes.

Finally, it is important to note that turbulence in the real atmosphere usually consists of many large positive and negative values of the instantaneous fluxes, such as heat flux $w'\theta'$. Only after averaging does a smaller, but significant, net flux $\overline{w'\theta'}$ become apparent.

An example of the instantaneous heat flux values measured by an aircraft flying near the ground during the 1983 Boundary Layer Experiment (BLX83, see Stull & Eloranta, 1984) is shown in Fig. 2.13. The net flux for this case was $\overline{w'\theta'} = 0.062\ \mathrm{K{\cdot}m{\cdot}s^{-1}}$. We see from the figure that most of the time there are small positive and negative values of $w'\theta'$ that average to near zero. Occasional large positive spikes associated with convective plumes cause the net average value of $\overline{w'\theta'}$ to be positive for this case. Figs 2.14 show the corresponding histograms of frequency of occurrence of w', θ', and $w'\theta'$. The aircraft was flying at about 75 m/s, so it is easy to convert the time axis of Fig 2.13 to a distance axis. Statistics for this afternoon case are shown in table 2-1.

Table 2-1. Statistics for the 100 s segment time series shown in Fig 2.14. This segment is extracted from a 4 min flight leg near the surface, during the BLX83 field experiment. The 4 min mean values were used as the reference from which the perturbation values were calculated, which explains why $\overline{w'}$ and $\overline{\theta'}$ are not exactly zero for this 100 s segment.

Statistic	w' (m/s)	θ' (K)	w'θ' (K m/s)
Average	0.017	-0.017	0.062
σ	0.67	0.18	0.14
Maximum	2.51	0.87	1.93
Minimum	-2.07	-0.44	-0.38

2.7.2 Turbulent Flux Profiles

Fig 2.15 shows idealizations of the turbulent heat, momentum, and moisture fluxes for both the daytime and nighttime BLs. During the daytime, the fluxes are large, and usually change linearly with height over the ML. At night, the fluxes are much weaker.

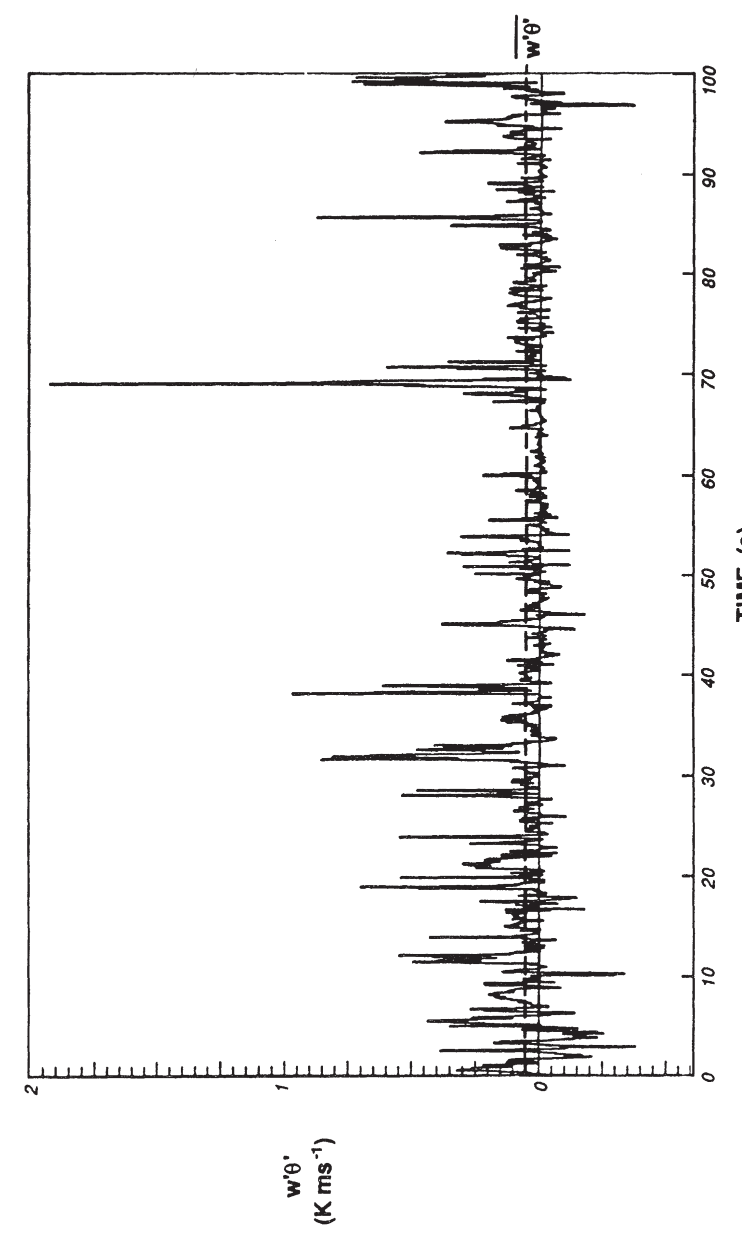


Fig. 2.13 Instantaneous kinematic surface heat flux. w'θ' trace measured by aircraft (BLX83 flight 3, leg 3) May 28, 1983 Oklahoma. The dashed line indicates the average heat flux.

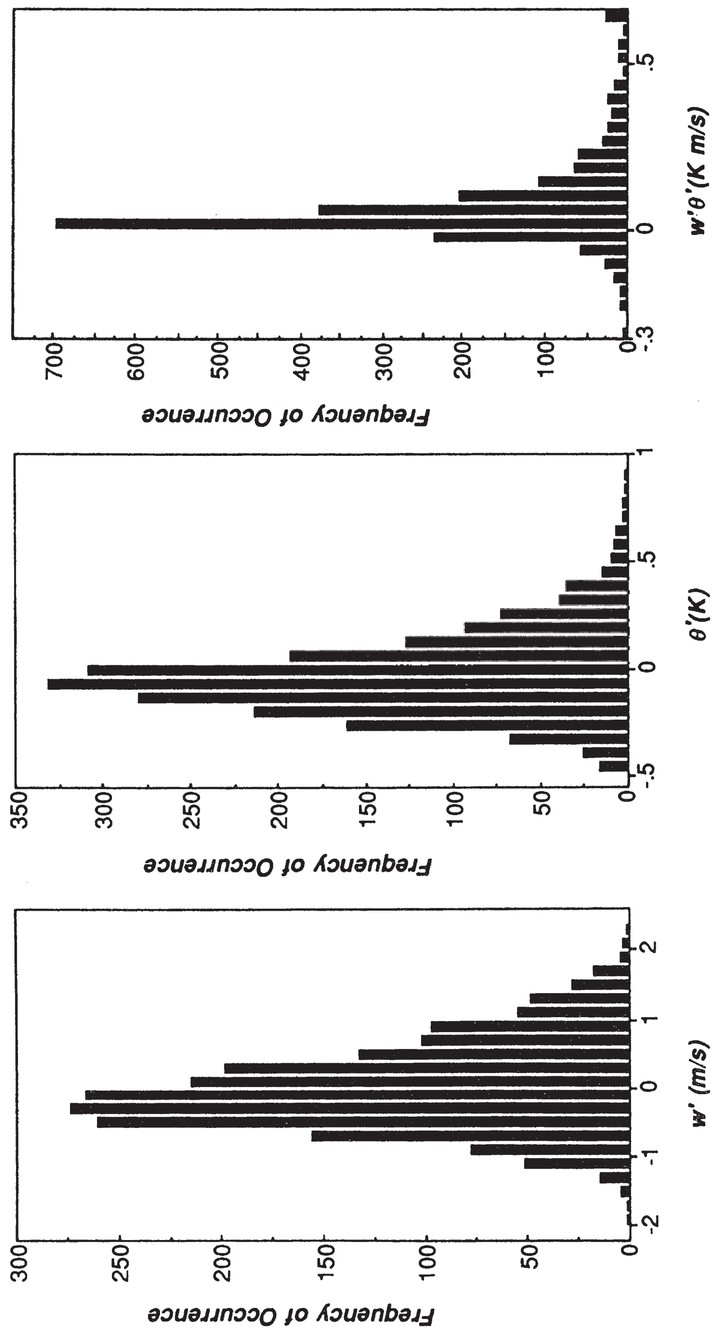


Fig. 2.14 Frequency of occurrence of w', θ', and w'θ'.

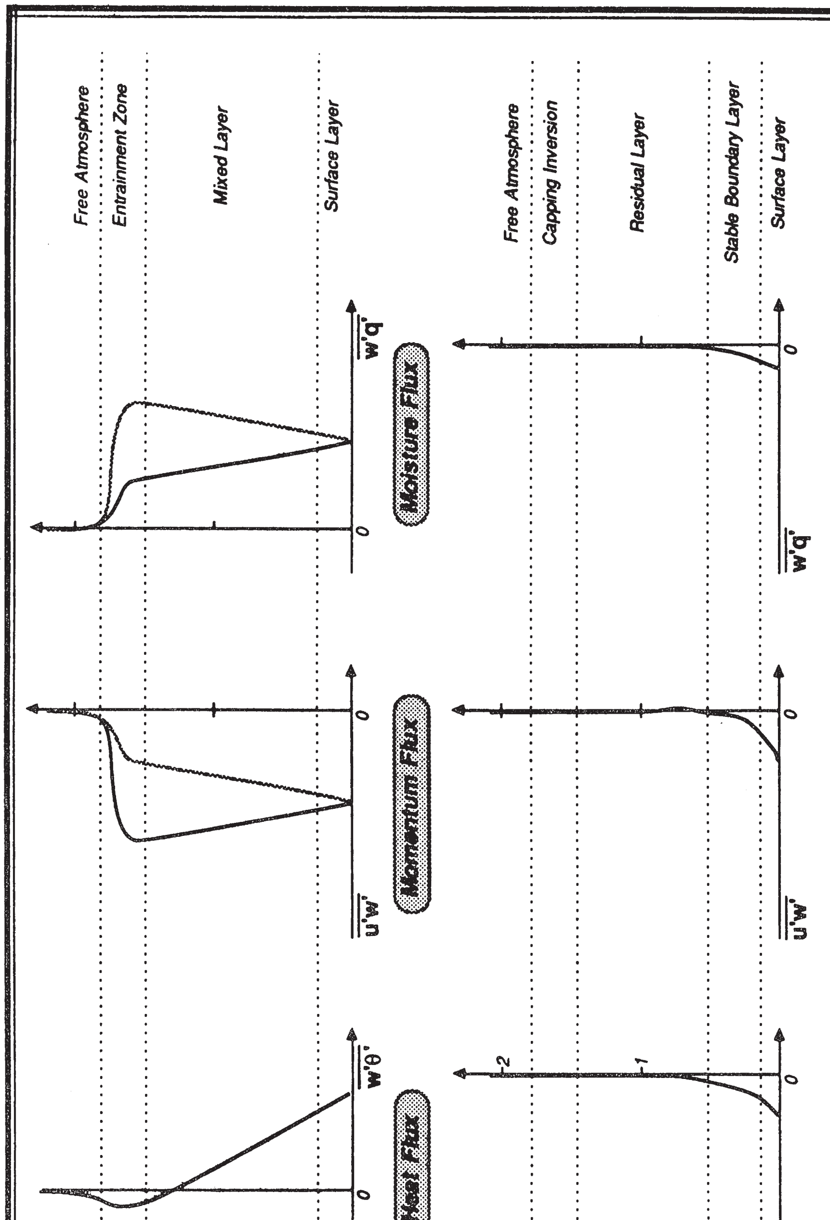


Fig. 2.15 Idealized vertical soundings of turbulent fluxes for the (a) convective mixed layer and (b) stable boundary layer.

2.8 Summation Notation

In the last section we encountered heat fluxes with three components and momentum fluxes with nine components. It is very laborious to write a separate forecast equation for each of these nine fluxes, but seemingly necessary if we want to better understand the boundary layer.

To ease the burden, we can employ a shorthand notation known as Einstein's summation notation. With just one term, we can represent all nine of the momentum fluxes. In this section, we first define some terms, then state rules with some examples, and finally show how summation notation and vector notation are related.

2.8.1 Definitions and Rules

Let m, n, and q be integer variable indices that can each take on the values of 1, 2, or 3. Let A_m represent a generic velocity vector, X_m represent a generic component of distance, and δ_m represent a generic unit vector (a vector of length unity and direction in one of the three Cartesian directions). By using indices as subscripts to these generic variables, we can define:

m = 1, 2, or 3	$A_1 = U$	$X_1 = x$
n = 1, 2, or 3	$A_2 = V$	$X_2 = y$
q = 1, 2, or 3	$A_3 = W$	$X_3 = z$

A variable with:	no free (unsummed) indices	= scalar
	1 free index	= vector
	2 free indices	= tensor

Unit vectors: $\delta_1 = \mathbf{i}$ $\delta_2 = \mathbf{j}$ $\delta_3 = \mathbf{k}$

Physically, we expect that some forces act in all directions while others might act in just one or two. To be able to isolate such directional dependence, we must define two new terms with unusual characteristics:

Kronecker Delta (a scalar quantity even though it has two indices):

$$\delta_{mn} = \begin{cases} +1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases} \tag{2.8.1a}$$

Alternating Unit Tensor (a scalar even though it has three indices):

$$\varepsilon_{mnq} = \begin{cases} +1 & \text{for } mnq = 123, 231, \text{ or } 312 \\ -1 & \text{for } mnq = 321, 213, \text{ or } 132 \\ 0 & \text{for any two or more indices alike} \end{cases} \tag{2.8.1b}$$

The unit vector, δ_m, and the Kronecker delta, δ_{mn}, can easily be confused. They represent distinctly different quantities that are not interchangeable. To help distinguish between these two quantities, remember that the Kronecker delta is a scalar and always has two subscripts, while the unit vector is a vector and always has just one subscript.

Three fundamental rules apply within summation notation: two concern repeated indices within any one term, and the other concerns nonrepeated (free) indices.

Rule (a): Whenever two identical indices appear in the same one term, it is implied that there is a sum of that term over each value (1, 2, and 3) of the repeated index.

Rule (b): Whenever one index appears unsummed (free) in a term, then that same index must appear unsummed in all terms in that equation. Hence, that equation effectively represents 3 equations for each value of the unsummed index. This insures that all terms are tensorally consistent with the other terms in the equation.

Rule (c): The same index cannot appear more than twice in one term.

2.8.2 Examples

Problem 1 and Solution, demonstrating Rule (a).

$$A_n \frac{\partial B_m}{\partial X_n} = A_1 \frac{\partial B_m}{\partial X_1} + A_2 \frac{\partial B_m}{\partial X_2} + A_3 \frac{\partial B_m}{\partial X_3}$$

$$= U \frac{\partial B_m}{\partial x} + V \frac{\partial B_m}{\partial y} + W \frac{\partial B_m}{\partial z}$$

Problem 2 and Solution, demonstrating Rule (a).

$$\begin{aligned} \delta_{2n} A_n &= \delta_{21} A_1 + \delta_{22} A_2 + \delta_{23} A_3 \\ &= 0 + A_2 + 0 \\ &= V \end{aligned}$$

The latter example leads to an important general conclusion:

$$\delta_{mn} A_n = A_m \qquad (2.8.2a)$$

namely, the Kronecker delta changes the index of A from n to m.

Problem 3, demonstrating Rule (b): Given the following equation, expand it:

$$A_m = B_m + \delta_{mn} C_n$$

Solution to 3. In each term is the same unrepeated index, m. Thus, this equation represents three equations:

$$\begin{cases} A_1 = B_1 + \delta_{1n} C_n \\ A_2 = B_2 + \delta_{2n} C_n \\ A_3 = B_3 + \delta_{3n} C_n \end{cases}$$

The last term in each equation has the Kronecker delta, which means we can use the general conclusion above to yield:

$$\begin{cases} A_1 = B_1 + C_1 \\ A_2 = B_2 + C_2 \\ A_3 = B_3 + C_3 \end{cases}$$

Problem 4, demonstrating all rules: One form of the equation of motion is written here in summation notation. For now, just accept this equation as a given example; we will discuss the physics of it in more detail in the next chapter. This equation employs repeated and nonrepeated indices, the Kronecker delta, the alternating unit tensor, and the stress tensor τ (to be discussed in the next section). Let both A and B represent velocities. This is quite a complex example, which you should study carefully:

$$\frac{\partial A_m}{\partial t} + B_n \frac{\partial A_m}{\partial X_n} = -\delta_{m3}\, g + f_c\, \varepsilon_{mn3}\, B_n - \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{mn}}{\partial X_n}\right] \qquad (2.8.2b)$$

Using the previous rules and definitions, we can step-by-step expand the shorthand equation above to discover the equivalent set of equations written in more conventional form.

Solution to 4. First, sum over repeated indices:

$$\frac{\partial A_m}{\partial t} + B_1 \frac{\partial A_m}{\partial X_1} + B_2 \frac{\partial A_m}{\partial X_2} + B_3 \frac{\partial A_m}{\partial X_3} = -\delta_{m3}\, g + f_c\, \varepsilon_{m13}\, B_1 + f_c\, \varepsilon_{m23}\, B_2$$

$$+ f_c\, \varepsilon_{m33}\, B_3 - \frac{1}{\rho}\frac{\partial p}{\partial X_m} + \frac{1}{\rho}\left[\frac{\partial \tau_{m1}}{\partial X_1} + \frac{\partial \tau_{m2}}{\partial X_2} + \frac{\partial \tau_{m3}}{\partial X_3}\right]$$

The term with ε_{m33} becomes zero because of the repeated index in the alternating unit tensor.

Next, write a separate equation for each value of the free index, m:

For m = 1:

$$\frac{\partial A_1}{\partial t} + B_1 \frac{\partial A_1}{\partial X_1} + B_2 \frac{\partial A_1}{\partial X_2} + B_3 \frac{\partial A_1}{\partial X_3} = -\delta_{13}\, g + f_c\, \varepsilon_{113}\, B_1 + f_c\, \varepsilon_{123}\, B_2 - \frac{1}{\rho}\frac{\partial p}{\partial X_1} + \frac{1}{\rho}\left[\frac{\partial \tau_{11}}{\partial X_1} + \frac{\partial \tau_{12}}{\partial X_2} + \frac{\partial \tau_{13}}{\partial X_3}\right]$$

In this equation the terms with δ_{13} and ε_{113} are both zero. We will leave out similar terms in the equations for the remaining components. The factor $\varepsilon_{123} = 1$.

For m = 2:

$$\frac{\partial A_2}{\partial t} + B_1 \frac{\partial A_2}{\partial X_1} + B_2 \frac{\partial A_2}{\partial X_2} + B_3 \frac{\partial A_2}{\partial X_3} = + f_c\, \varepsilon_{213}\, B_1 - \frac{1}{\rho}\frac{\partial p}{\partial X_2} + \frac{1}{\rho}\left[\frac{\partial \tau_{21}}{\partial X_1} + \frac{\partial \tau_{22}}{\partial X_2} + \frac{\partial \tau_{23}}{\partial X_3}\right]$$

The factor ε_{213} in the equation above equals -1.

For m = 3:

$$\frac{\partial A_3}{\partial t} + B_1 \frac{\partial A_3}{\partial X_1} + B_2 \frac{\partial A_3}{\partial X_2} + B_3 \frac{\partial A_3}{\partial X_3} = -\delta_{33}\, g - \frac{1}{\rho}\frac{\partial p}{\partial X_3} + \frac{1}{\rho}\left[\frac{\partial \tau_{31}}{\partial X_1} + \frac{\partial \tau_{32}}{\partial X_2} + \frac{\partial \tau_{33}}{\partial X_3}\right]$$

The factor $\delta_{33} = 1$ in the equation above.

After substituting U for A_1, y for X_2, τ_{zx} for τ_{31}, etc., we finally get:

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + W\frac{\partial U}{\partial z} = +f_c V - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left[\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right]$$

$$\frac{\partial V}{\partial t} + U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + W\frac{\partial V}{\partial z} = -f_c U - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\left[\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right]$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial x} + V\frac{\partial W}{\partial y} + W\frac{\partial W}{\partial z} = -g - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left[\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right] \tag{2.8.2c}$$

Discussion of Problem 4. Comparing the above set of equations to the original shorthand version, we begin to appreciate the power of Einstein's summation notation. This tool will be used throughout the remaining chapters. It is analogous to vector notation, which is examined in the next section.

Usually, the shorthand (summation) form of (2.8.2b) is written as follows:

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\delta_{i3}\, g + f_c\, \varepsilon_{ij3}\, U_j - \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\left[\frac{\partial \tau_{ij}}{\partial x_j}\right] \tag{2.8.2d}$$

where vectors like U_i have three components (U, V, W), and where (i, j, k) are indices, not unit vectors. This is the form that will be used for the remainder of the text.

Finally, we should recognize that a whole term is a scalar, vector, or tensor if the term has zero, one, or two unsummed (free) variable indices, respectively. For example, the $f_c\, \varepsilon_{ij3}\, U_j$ term in the equation above is a vector, because there is only one unsummed variable index, i.

2.8.3 Comparison with Vector Notation

Vectors can represent three Cartesian components, and tensors can represent nine. There is a one-to-one correspondence between vector definitions and Einstein's summation notation, as might be expected. The following is an explanation of how vectors can be rewritten in summation notation, and how vector operations such as the dot and cross product can be represented. In these examples, vector operations apply only to the vector parts of each term; scalar parts can be separated and interpreted as simple products.

The definitions below relate basic vector forms (especially, unit vectors) and operators to summation notation:

Vector: $$\mathbf{A} \equiv A_m \, \delta_m \tag{2.8.3a}$$

Dot Product: $$\delta_m \bullet \delta_n \equiv \delta_{mn} \tag{2.8.3b}$$

Cross Product: $$\delta_m \times \delta_n \equiv \varepsilon_{mnq} \, \delta_q \tag{2.8.3c}$$

Del Operator: $$\nabla(\;) \equiv \delta_m \frac{\partial(\;)}{\partial X_m} \tag{2.8.3d}$$

Examples are presented here of other vector operations using the definitions above. First, consider the dot product between two vectors:

$$\begin{aligned}
\mathbf{A} \bullet \mathbf{B} &= (\delta_m \, A_m) \bullet (\delta_n \, B_n) \\
&= (\delta_m \bullet \delta_n) \, A_m \, B_n \\
&= \delta_{mn} \, A_m \, B_n \\
&= A_m \, B_m
\end{aligned} \tag{2.8.3e}$$

On the first line, we substituted each vector by its summation notation as three Cartesian components times their respective component magnitudes. On the second line, the vector (boldface) terms were grouped together, leaving the product of the magnitudes remaining at the end. Then, the definition of a vector dot product was used to substitute the Kronecker delta. Finally, the Kronecker delta was used to change one subscript to equal the other. The end result, $A_m B_m = A_1 B_1 + A_2 B_2 + A_3 B_3$, is indeed a scalar that is equal in value to the scalar result of the vector dot product.

A similar development can be made for the cross product of two vectors:

$$\begin{aligned}
\mathbf{A} \times \mathbf{B} &= (\delta_m \, A_m) \times (\delta_n \, B_n) \\
&= (\delta_m \times \delta_n) \, A_m \, B_n \\
&= \varepsilon_{mnq} \, A_m \, B_n \, \delta_q
\end{aligned} \tag{2.8.3f}$$

The result is a vector, as is required for a cross product between two vectors. The reader can perform the implied sums to verify that the expected terms are obtained. Note that $\varepsilon_{mnq} = \varepsilon_{qmn} = \varepsilon_{nqm}$, resulting in equivalent but different-looking expressions for (2.8.3f).

As a final example, we will look at the divergence of a vector:

$$
\begin{aligned}
\nabla \bullet A &= \left(\delta_m \frac{\partial}{\partial X_m}\right) \bullet (\delta_n A_n) \\
&= (\delta_m \bullet \delta_n) \frac{\partial A_n}{\partial X_m} \\
&= (\delta_{mn}) \frac{\partial A_n}{\partial X_m} \\
&= \frac{\partial A_m}{\partial X_m}
\end{aligned}
\qquad (2.8.3g)
$$

We will have little use for vector notation in the remainder of this book, because summation notation is frequently easier to use. This section was presented only because most meteorologists are familiar with vector notation from their studies of atmospheric dynamics.

2.9 Stress

We have seen that the covariance statistic describes a turbulent flux. But a momentum flux is analogous to a stress. In this section, we review the nature of stress and relate it to various turbulence statistics.

Stress is the force tending to produce deformation in a body. It is measured as a force per unit area. Three types of stress appear frequently in studies of the atmosphere: pressure, Reynolds stress, and viscous shear stress.

2.9.1 Pressure

Pressure is a type of stress that can act on a fluid at ***rest***. For an infinitesimally small fluid element, such as idealized as the cube sketched in Fig 2.16a, pressure acts equally in all directions. ***Isotropic*** is the name given to characteristics that are the same in all directions (see Figs 2.4 and 2.5).

If we consider just one face of this cube, as in Fig. 2.16b, we see that the isotropic nature of pressure tends to counteract itself in all directions except in a direction normal to (perpendicular to) the surface of the cube. Forces acting normal to all faces of the cube tend to compress or expand the cube, thereby deforming it (Fig 2.16c).

At sea level, the standard atmospheric pressure is 1.013×10^5 N/m^2. A pascal (Pa) is defined as 1 N/m^2, thus standard atmospheric pressure at sea level is 101.3 kPa. Historically, millibars (mb) have also been used as a pressure unit, where 1 mb = 100 pascals. In kinematic units, standard sea-level pressure is 82714 m^2/s^2. Although this value is much larger than the other stresses to be discussed next, it is almost totally counteracted by the influence of gravity, as described by the hydrostatic approximation.

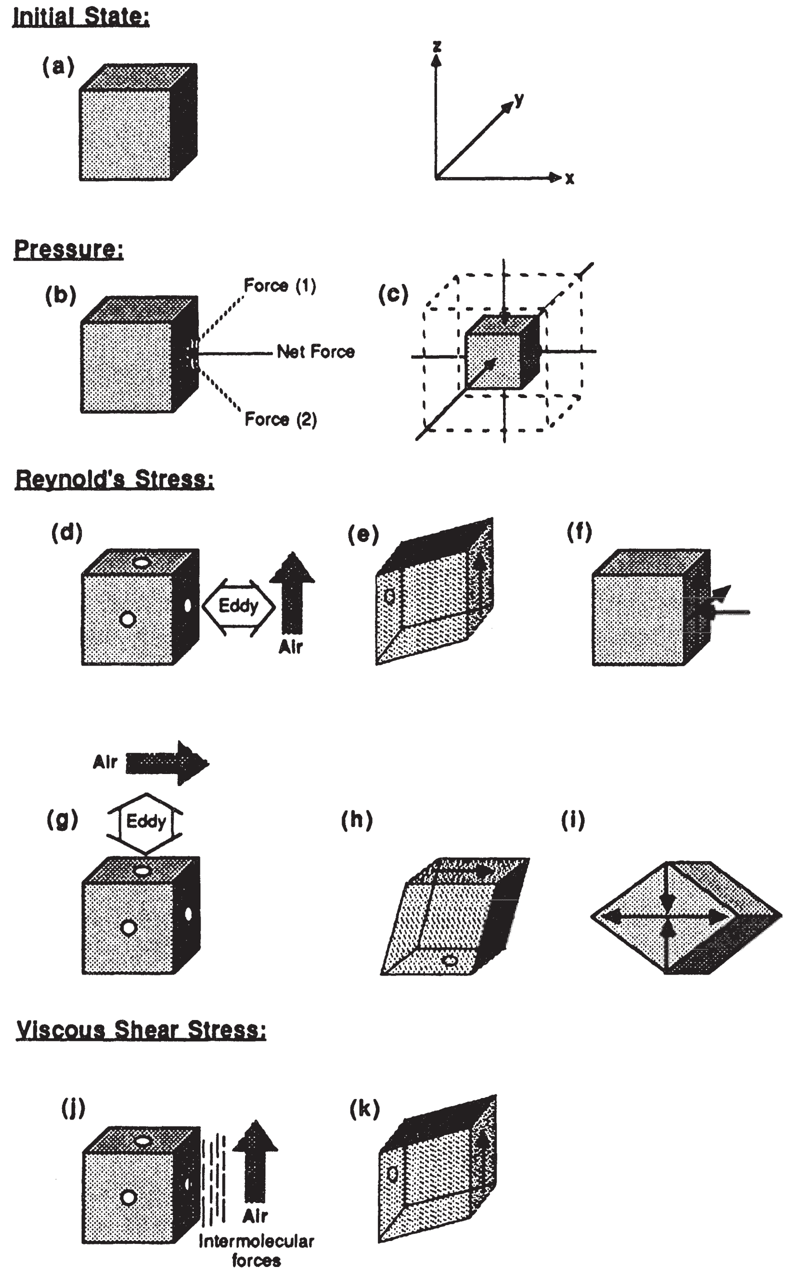


Fig. 2.16 Sketch of the effects of stress on a conceptual cube of fluid.

Pressure is a concept that is familiar to most meteorologists; therefore, we will not dwell on it here. Because pressure is not dependent on direction, we need just one number to describe it at any point in space and time. Thus, pressure is a scalar.

2.9.2 Reynolds stress

Reynolds stress exists only when the fluid is in ***turbulent motion***. A turbulent eddy can mix air of different wind speeds into our cube of interest (Fig 2.16d). When this different-speed air is incorporated into one face of the cube and not the opposite face, the cube deforms because of the velocity differences between those two faces (Fig 2.16e).

The rate that air of different speeds is transported across any face of the cube is just the momentum flux, by definition (see section 2.6). The effect of this flux on the cube is identical to what we would observe if we applied a force on the face of the cube; namely, the cube would deform. Thus, turbulent momentum flux acts like a stress, and is called the ***Reynolds stress***. In this example, air moving upward (possessing w'), was mixed towards the cube (at rate u'), resulting in a Reynolds stress component described by $-\rho\,\overline{u'w'}$. The magnitude of this component of Reynolds stress or momentum flux in kinematic units is thus $|\overline{u'w'}|$. Sometimes the symbol τ_{Reynolds} is used for the Reynolds stress.

Even if we consider just one face of the cube (Fig 2.16f), air moving in any one of the three Cartesian directions could be mixed into it, resulting in a variety of deformations. For that one face, we must consider $\overline{u'u'}$, $\overline{u'v'}$, and $\overline{u'w'}$. Since the same number of combinations holds for faces normal to the other two cartesian directions, we have a total of nine components of the Reynolds stress to account for. Based on what we learned in section 2.6, this is expected because there are nine components of the momentum flux.

Let's consider one other example. Suppose a u' wind gust is being turbulently mixed towards the top face of a cube at rate w', as is sketched in Fig 2.16g. The resulting deformation shown in Fig 2.16h is associated with a $\overline{w'u'}$ momentum flux. The nature of the deformation shown in Fig 2.16h is identical to that of Fig 2.16e; the only difference is that (e) is rotated relative to the orientation of (h). Based on the nature of the deformation alone, we must then conclude that $\overline{u'w'} = \overline{w'u'}$. Fig 2.16i shows the type of deformation that is exhibited by both Figs 2.16 e and h, without regard to the rotational differences.

Similar arguments can be made for the other faces. We can conclude that the Reynolds stress tensor (represented by the elements in the matrix below) is symmetric:

$$\begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{v'u'} & \overline{v'v'} & \overline{v'w'} \\ \overline{w'u'} & \overline{w'v'} & \overline{w'w'} \end{bmatrix} = \begin{bmatrix} \overline{u'u'} & \overline{u'v'} & \overline{u'w'} \\ \overline{u'v'} & \overline{v'v'} & \overline{v'w'} \\ \overline{u'w'} & \overline{v'w'} & \overline{w'w'} \end{bmatrix} \tag{2.9.2}$$

Hence, we need only be concerned with six independent stress components. The convenience of Einstein's summation notation is apparent here. We could easily represent any one of the components of the matrix on the left by $\overline{u_i' u_j'}$. The kinematic Reynolds stress in typical atmospheric surface layers is on the order of 0.05 m^2/s^2.

Momentum flux and Reynolds stress are properties of the ***flow***, not of the fluid. This stress is fully described by the matrices above, which contain products of velocities (a flow characteristic) that could apply to any fluid. Such is not the case with viscous shear stresses. Although the effect of $\overline{u_i' u_j'}$ is like a stress, the Reynolds stress is not a true stress (force per unit area) as is the viscous shear stress.

2.9.3 Viscous Stress

Viscous stress exists when there are ***shearing motions*** in the fluid. The motion can be laminar or turbulent. When one portion of a fluid moves, the intermolecular forces tend to drag adjacent fluid molecules in the same direction (Fig 2.16j). The strength of these intermolecular forces depend on the nature of the fluid: molasses has stronger forces than water, which in turn has stronger forces than air. A measure of these forces is the viscosity. The result of this stress is a deformation of the fluid (Fig. 2.16k).

These viscous forces can act in any of the three cartesian directions on any of the three faces of our conceptual cube (Fig. 2.16f). Thus, the viscous stress is also a tensor with nine components. Like the Reynolds stress, the viscous-stress tensor is symmetric, leaving six independent components.

A fluid for which the viscous stress is linearly dependent on the shear is said to be a ***Newtonian fluid***. The stress, τ_{ij}, for a Newtonian fluid is usually given by:

$$\tau_{ij} = \mu \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) + \left(\mu_B - \frac{2}{3}\mu \right) \frac{\partial U_k}{\partial x_k} \delta_{ij} \tag{2.9.3}$$

where μ_B is the bulk viscosity coefficient (near zero for most gases) and μ is the dynamic viscosity coefficient. We can interpret τ_{ij} as the force per unit area in the x_i-direction acting on the face that is normal to the x_j-direction.

The viscous stress can be put into kinematic form by dividing by the mean density of the fluid. A corresponding kinematic viscosity is defined by $\nu = \mu / \rho$. The standard atmospheric sea-level value for kinematic viscosity of air is 1.4607×10^{-5} m^2/s. For a mean wind shear of 0.5 s^{-1} (typical for atmospheric surface layers), the resulting viscous stress is 7.304×10^{-6} m^2/s^2.

This value is so much smaller than the Reynolds stresses in the BL that the viscous stress is usually neglected in mean wind forecasts. Turbulent eddies, however, can have much larger values of shear in localized eddy-size regions. We thus can not neglect viscosity when forecasting turbulence.

2.10 Friction Velocity

During situations where turbulence is generated or modulated by wind shear near the ground, the magnitude of the surface Reynolds' stress proves to be an important scaling variable. The total vertical flux of horizontal momentum measured near the surface is

$$\tau_{xz} = -\bar{\rho}\,\overline{u'w'}_s \qquad \text{and} \qquad \tau_{yz} = -\bar{\rho}\,\overline{v'w'}_s$$

$$\left|\tau_{\text{Reynolds}}\right| = \left[\tau_{xz}^{\;2} + \tau_{yz}^{\;2}\right]^{1/2} \tag{2.10a}$$

Based on this relationship, a velocity scale called the ***friction velocity***, u_*, is defined as

$$u_*^2 \equiv \left[\overline{u'w'}_s^{\;2} + \overline{v'w'}_s^{\;2}\right]^{1/2}$$

$$= \left|\tau_{\text{Reynolds}}\right| / \bar{\rho} \tag{2.10b}$$

For the special case where the coordinate system is aligned so that the x-axis points in the direction of the surface stress, we can rewrite (2.10b) as $u_*^2 = \left|\overline{u'w'}_s\right| = |\tau_{\text{Reynolds}}| / \bar{\rho}$. Examples of u_* evolution are shown in Figs 2.17 and 4.1.

While we are talking about surface scales, we can also introduce surface layer temperature (θ_*^{SL}) and humidity (q_*^{SL}) scales that are defined by:

$$\theta_*^{SL} = \frac{-\overline{w'\theta'_s}}{u_*} \tag{2.10c}$$

$$q_*^{SL} = \frac{-\overline{w'q'_s}}{u_*} \tag{2.10d}$$

These scales will be used in later chapters dealing with surface-layer similarity theory.

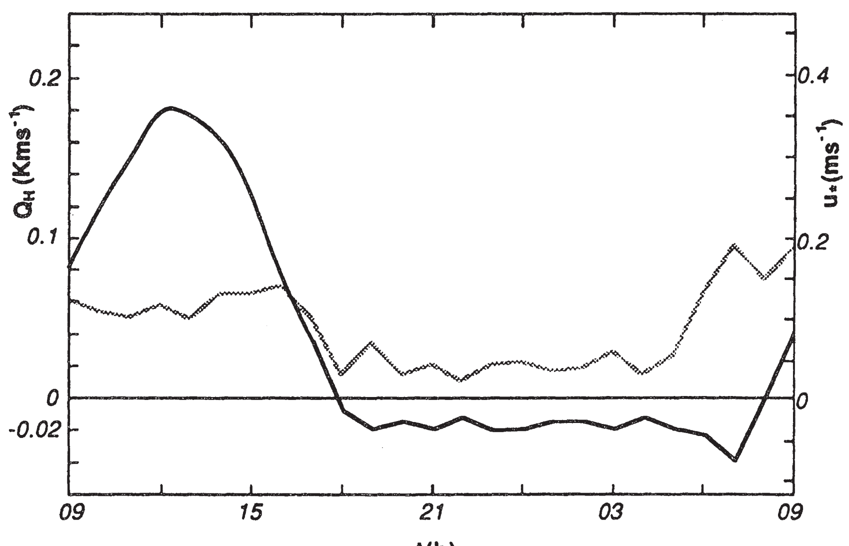


Fig. 2.17 Sample of diurnal variation of kinematic heat flux Q_H (solid line) and friction velocity u_* (shaded line), after Andre, et al., (1978).

2.11 References

André, J.-C., G. De Moor, P. Lacarrère, G. Therry, and R. du Vachat, 1978: Modeling the 24-hour evolution of the mean and turbulent structure of the planetary boundary layer. *J. Atmos. Sci.*, **35**, 1861-1883.

Bean, B.R. & T.P Repoff, 1983: A study of turbulent energy over complex terrain (STATE, 1978). *Bound.-Layer Meteor.*, **25**, 17-23.

Caughey, S.J., J.C. Wyngaard and J.C. Kaimal, 1979: Turbulence in the evolving stable boundary layer. *J. Atmos. Sci.*, **36**, 1041-1052.

Richardson, L.F., 1922: *Weather Prediction by Numerical Process.* Cambridge Univ. Press.

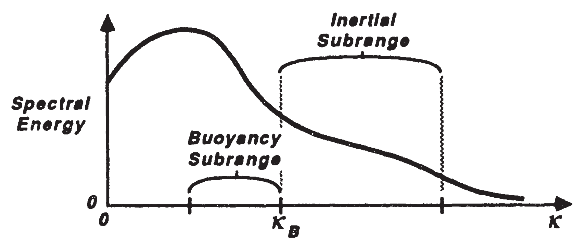
Stull, R.B. and E.W. Eloranta, 1984: Boundary layer experiment - 1983. *Bull. Am. Meteor. Soc.*, **65**, 450-456.

Van der Hoven, I., 1957: Power spectrum of horizontal wind speed in the frequency range from 0.0007 to 900 cycles per hour. *J. Meteor.*, **14**, 160.

Van Dyke, M., 1982: *An Album of Fluid Motion.* The Parabolic Press, Stanford.176 pp.

2.12 Exercises

1) It has been suggested that in regions of strong static stability, the lower (long wavelength, small wavenumber) end of the inertial subrange occurs at a wavenumber, κ_B, given by $\kappa_B \cong N_{BV}^{3/2}\,\varepsilon^{-1/2}$, where N_{BV} is the Brunt-Vaisala frequency, and ε is the turbulence dissipation rate. Between this wavenumber and lower wavenumbers is a region called the buoyancy subrange, where the gravitational effects (i.e., buoyancy) are important. Within the buoyancy subrange sketched below, would you expect turbulence to be isotropic?



2) Given the following instantaneous measurements of potential temperature (θ) and vertical velocity (w) in this table, fill in all the remaining blanks in the table. Also, verify with the answers from above that $\overline{w\theta} = \overline{w}\,\overline{\theta} + \overline{w'\theta'}$.

	Measurements:		Calculations:					
index	w	θ	w'	θ'	(w')^2	(θ')^2	wθ	w'θ'
0	0.5	295						
1	-0.5	293						
2	1.0	295						
3	0.8	298						
4	0.9	292						
5	-0.2	294						
6	-0.5	292						
7	0.0	289						
8	-0.9	293						
9	-0.1	299						
Average:								

3) Given the data in problem (2), find the biased standard deviation for w and θ, and find the linear correlation coefficient between w and θ.

4) Using your results from problems (2) and (3), is the data characteristic of a stable, neutral, or unstable boundary layer?

5) Let: c = constant, s $\neq$ function of time, and $A = \overline{A} + a'$, $B = \overline{B} + b'$, and $E = \overline{E} + e'$.
Expand the following terms into mean and turbulent parts, and apply Reynold's averaging rules to simplify your expression as much as possible:

a) $\overline{(c\,A\,B)} = ?$

b) $\overline{(A\,B\,E)} = ?$

c) $\overline{\left(A \cdot \frac{\partial B}{\partial s}\right)} = ?$

d) $\overline{\left(\frac{\partial A}{\partial s}\right) \cdot \left(\frac{\partial B}{\partial s}\right)} = ?$

e) $\overline{\left(c\ \nabla^2 A\right)} = ?$

6) The following terms are given in summation notation. Expand them (that is, write out each term of the indicated sums).

a) $\frac{\partial(\overline{u_i'\,u_j'})}{\partial x_j}$

b) $u_i' \frac{\partial \theta'}{\partial x_i}$

c) $\overline{U_j} \frac{\partial(\overline{u_i'\,u_k'})}{\partial x_j}$

d) $\overline{u_i'\,u_j'}\ \frac{\partial \overline{U_k}}{\partial x_j}$

e) $\frac{\partial(\overline{u_i'\,u_j'\,u_k'})}{\partial x_j}$

f) $\left(\frac{\partial u_i}{\partial x_j}\right)\left(\frac{\partial u_k}{\partial x_j}\right)$

7) Express the following terms in summation notation. They are given to you in vector notation.

a) Gradient of a scalar: ∇s

b) Curl of a vector: $\nabla \times V$

c) Total derivative: dV/dt

d) Laplacian: ∇^2

8) Consider a 100 m thick layer of air at sea level, with an initial potential temperature of 290 K. If the kinematic heat flux into the bottom of this layer is 0.2 K m/s and the flux out of the top is 0.1 K m/s, then what is the potential temperature of that layer 2 hours later? Assume that the potential temperature is constant with height in the layer. (This is a thought question, meant to stimulate the student's ability to interpret a physical situation in an acceptable mathematical framework.)

9) Given the typical variation of wind speed with height within the surface layer (see Chapt 1), and using a development similar to that in section 2.7:

a) determine whether the net kinematic momentum flux, $\overline{u'w'}$ is positive, negative, or zero within the surface layer.
b) Does you answer mean the momentum is being transported up or down, on the average?
c) This momentum that is transported up or down, where does it go or where does it come from, and how would that alter the mean state of the atmosphere?

10) Suppose we define an average wind speed by

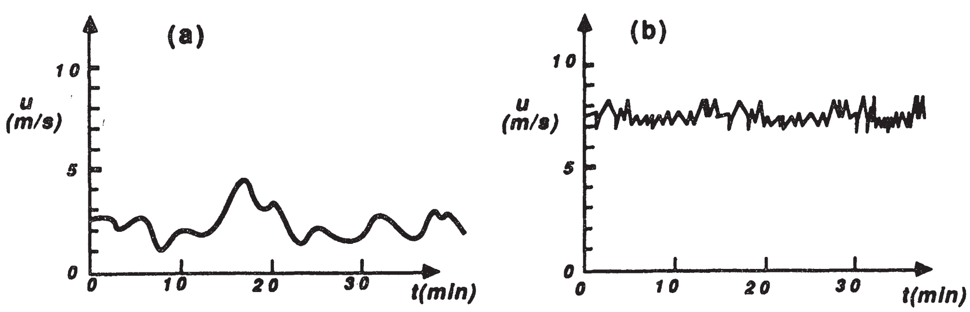
$$\overline{U}(P) = \frac{1}{N}\sum_{i=0}^{N-1} U(t_i)$$

where $t_0 = 0$, $t_N = P$, and where the averaging time, P, is the interval between t=0 and t=P. When P is small, we will find one value for $\overline{U}$, and when P is larger, we might find a different value. In fact, $\overline{U}$ (P) is probably a smoothly varying function of P. Given the instantaneous measurements of U(t) shown in the table below:
a) Plot U vs t.
b) Plot $\overline{U}$ vs P for P = 0 to 60 min. (A calculator or computer might make the job easier.)
c) Comment on the relationship between the spectral gap and your answer from part (b).

Data:

i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t (min)	0	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60
U (m/s)	3	4	3	6	6	2	5	3	7	8	5	4	3	5	2	2	4	5	3	6	5

11) Suppose that the following wind speed traces were observed on different days using an anemometer mounted on a 10m mast:



a) Sketch (both on the same graph) your best guess of the frequency spectrum of a) and of b). Do not digitize the data from graphs a) and b), but visually estimate the contribution of various size eddies to the total turbulence kinetic energy when answering this question.

b) If the mean wind speed at anemometer height was 5 m/s, then reanswer part a) in terms of a wavenumber spectrum rather than a frequency spectrum.

12) For selected operational weather forecast models, determine whether the grid spacing (for finite-difference models) or the smallest wavelength (for spectral models) falls within the spectral gap. Find out how the subgrid-scale motions (turbulence) are parameterized in those models. (Hint: The instructor should select one or two current models for this project. It involves outside reading on the part of the students.)

13) When might the ergodic assumption fail for boundary layer studies?

14) Turbulence is usually anisotropic, inhomogeneous, and nonstationary in the boundary layer, although we often limit our studies to situations where one or more of these special cases are approximately valid. Comment on the difference in meaning between the words isotropic, homogeneous, and stationary. Can turbulence be homogeneous but anisotropic? Can turbulence be isotropic but inhomogeneous? How are homogeniety and stationarity related?

15) What would be the dimensions of kinematic pressure flux? Of kinematic vorticity flux?

16) The solar constant is about 1380. W/m^2. Rewrite this in kinematic units.

17) What is the difference between $\partial U_j / \partial x_j$ and $\partial U_k / \partial x_k$?

18) If $\partial U_k / \partial x_k = 0$ (valid for incompressible flow), then write the resulting equation for $\partial \tau_{ij} / \partial x_j$ in the simplest form possible. [Hint, use what you learned from question (17) .]

19) Given the typical value for kinematic viscosity from section 2.9, find what value of shear is necessary to make the viscous shear stress as large as the previously quoted typical value of Reynolds stress. Could such a shear really be found within eddies? If so, what size eddies? (Hint, assume the viscous shear stress is equal to the viscosity times the shear).

20) How is the pressure stress different from a stress like $\overline{u' u'}$ or τ_{22} ?

21) Suppose that on the planet Krypton turbulent motions are affected by a strange form of viscosity that dissipates turbulent energy in only the vertical direction. How would the average turbulence kinetic energy differ from that found on earth?

22) How, if at all, are terms A and B below related to each other. Answer with equations or words.

$$A = f \varepsilon_{ij3} \overline{U_j} \quad , \qquad B = f \varepsilon_{ij3} \overline{U_i}$$

23) Given the term $U \partial(V^2) / \partial x$, which represents the U-advection of the V-component of total kinetic energy. Expand the U and V variables into mean and turbulent parts, Reynolds average, and simplify as much as possible.

24) Expand the Coriolis term [$-2\ \varepsilon_{ijk}\ \Omega_j\ U_k$] using summation notation for the case i=1. Assume that Ω_j = (0, ω cos λ, ω sin λ) are the three components of the angular velocity vector, where λ is latitude and ω is the speed of rotation of the earth (ω = 360 degrees in 24 hours). Assume that W = 0 for simplicity.

25) For each separate term in problem 6, count the number of indices and determine if each term is a scalar, vector, tensor, etc.

26) Given the following variances in m^2/s^2:

Where:	Location A		Location B	
When (UTC):	1000	1100	1000	1100
$\overline{u'^2}$	0.50	0.50	0.70	0.50
$\overline{v'^2}$	0.25	0.50	0.25	0.25
$\overline{w'^2}$	0.70	0.50	0.70	0.25

Where, when and for which variables is the turbulence:

a) Stationary?

b) Homogeneous?

c) Isotropic?

27) What boundary layer flow phenomena or characteristics have scale sizes on the order of:

a) 1 mm ?

b) 10 m ?

c) 1 km ?

28) Simplify the following term (assume horizontal homogeneity).

$$\delta_{k1}\ \varepsilon_{ijk}\ \frac{\partial\ \overline{u_i'\ \theta'}}{\partial x_j}$$