ATM 505 – Module 4 – Homework Answers

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1. (a) A 100nm (0.1μm) diameter NaCl particle corresponds to the middle solid Köhler curve in the provided figure. The diameter of the droplet that will form on this NaCl particle exposed to a water vapor supersaturation of 0.1% (corresponding to a relative humidity of 100.1%) is **∼ $1\mu m$** *[5 pts]* . The droplet would not spontaneously grow larger until the supersaturation of the surrounding air exceeded ∼0.12%.
   [5 pts]

   (b) A 50nm (0.05μm) diameter $(NH_4)_2SO_4$ particle corresponds to the uppermost dashed Köhler curve in the provided figure. The relative humidity of the air surrounding a 300nm (0.3 μm) droplet that follows this curve must be ∼100.4% *[5 pts]* (or 0.4% supersaturation). If the supersaturation were to rise slightly further (to >∼ 0.45%), the droplet would become "activated" and begin growing unstably.
   [5 pts]

   (c) Again, a 50nm (0.05μm) diameter $(NH_4)_2SO_4$ particle corresponds to the uppermost dashed Köhler curve in the provided figure. As noted in (b), the droplet would begin to grow unstably as supersaturation increased above **0.45%** *[5 pts]*. This marks the transition from a haze droplet to a cloud droplet and defines the particle's critical supersaturation.
   [5 pts]

   (d) The dry particle mass m_{dry}, assuming the particle is spherical, can be calculated as $m_{dry} = \rho_s(\pi/6)d_p^3$, where ρ_s is the dry particle density (which is 1.77 g/cm^3 for the non-volatile solute $(NH_4)_2SO_4$) and $(\pi/6)d_p^3$ is the volume for a sphere with diameter d_p. For an $(NH_4)_2SO_4$ particle with $d_p = 50$nm, **$m_{dry} \sim 10^{-10}\mu g$** *[2.5 pts]*.
   To calculate the water mass m_{H2O} the volume of water must be inferred from the difference in the droplet total volume (H_2O + $(NH_4)_2SO_4$) and the dry $(NH_4)_2SO_4$ volume (Fig. 1), i.e.:

$$m_{H2O} = \rho_L(\pi/6)(D_p^3 - d_p^3)$$

where ρ_L is the liquid water density and D_p is the droplet diameter. With $D_p = 300$nm, $d_p = 50$nm and $\rho_L = 1$ g/cm^3, $\boldsymbol{m_{H2O} \sim 10^{-8}\mu g}$ [2.5 pts]. To calculate the water mass fraction simply evaluate:

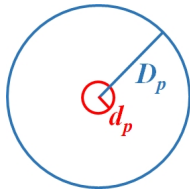$$m_{H2O}/(m_{dry} + m_{H2O}) \sim \boldsymbol{\underline{99\%}}$$

Figure 1: Water volume can be estimated by determining the total volume $(\pi/6)D_p^3$ and subtracting the volume of the dry residual particle $(\pi/6)d_p^3$ (i.e. what's left behind after the water evaporates).

[5 pts]
[20 pts total]

There are ~500,000 sulfate and ~1 million ammonium ions in solution, and ~500 million water molecules in this 300nm haze droplet. [extra credit up to 6 pts]

2. (a) The goal of this question is to solve for the condensational growth rate dD_p/dt for a pure water droplet of 80nm at 101% RH.

Since this is a pure water droplets, i.e. no solute, first calculate S_{eq} using the Kelvin equation:

$$S_{eq} = \frac{e_s(T, D_p)}{e_s(T)} = \exp\left[\frac{4\sigma}{D_p R T n_L}\right]$$
$$= \exp\left[\frac{4 * (72.7 dyn/cm) * (10^6 \mu m/100 cm)}{(0.08 \mu m) * (8.31x10^7 dyn \cdot cm/K \cdot mol) * (303K) * (0.056 mol/cm^3)}\right]$$
$$= \boldsymbol{\underline{1.026}} [\boldsymbol{5 \ pts}]$$

As shown in the lecture, the droplet growth equation can be written in terms of the difference in either vapor pressure or saturation ratio:

$$\frac{dD_p}{dt} = \frac{4D_v\left[e_\infty - e_s(T, D_p, x_s)\right]}{D_p R T n_L}$$
$$= \frac{4D_v\left[e_s(T) \cdot (S_\infty - S_{eq})\right]}{D_p R T n_L},$$

since $S_{eq} = e_s(T, D_p)/e_s(T)$ for a pure water droplet and $S_\infty = e_\infty/e_s(T)$. The difference **$S_\infty - S_{eq}$= 1.01-1.026***[5 pts]* = -0.016 is the driving force for droplet growth. In this case, since $S_\infty < S_{eq}$, the droplet is evaporating.

Since $S_\infty - S_{eq}$ has already been determined, all that remains to be calculated before solving for dD_p/dt is the saturation vapor pressure $e_s(T)$ from the Clausius-Clapeyron equation.

$$e_s(303K) = 6.11mb \cdot \exp\left[\frac{2.5\text{x}10^6 \cancel{J/kg}}{461 \cancel{J/kgK}}\left(\frac{1}{273\cancel{K}} - \frac{1}{303\cancel{K}}\right)\right] = \mathbf{\underline{43.7mb}}\textit{[2.5 pts]}$$

Now we can plug $e_s(T)$ and $S_\infty - S_{eq}$ in to the droplet growth equation to calculate the droplet evaporation rate:

$$\frac{dD_p}{dt} = \frac{4*(0.24cm^2/s)\left[(43.7\cancel{mb})*(-0.016)\right]*(10^6\cancel{\mu m}/100\cancel{cm})*(100\cancel{J/m^3mb})}{(0.08\cancel{\mu m})*(8.31\cancel{J/molK})*(303\cancel{K})*(0.056\cancel{mol/cm^3})*(10^6\cancel{cm^3/m^3}))}$$
$$= \mathbf{\underline{-0.06cm/s}}\textit{[2.5 pts]}$$

Since the droplet is only 80nm = 0.000008 cm, the droplet will evaporate completely in far less than one second!
[15 pts]

(b) A pure water droplet is not expected to exist at a stable size: It will be either growing by condensation or evaporating, depending on its size and the ambient relative humidity. Although a supersaturation of 1% (RH = 101%) is about as high as you would expect a droplet to experience in Earth's atmosphere, a pure water droplet could be evaporating at even this high humidity. As shown in the calculations in 2a, a pure water droplet of diameter 80nm would quickly evaporate in a 1% supersaturation environment. The Köhler equation shows that **addition of solute (x_s) decreases S_{eq}***[2.5 pts]*. If enough solute were added, that could reduce S_{eq} enough to compensate for the increased internal pressure of the droplet such that the droplet would grow at 1% supersaturation instead of evaporating. For that to occur, the additional solute would have to bring S_{eq} below 1.01. Two scenarios are possible with addition of solute. Droplets would: **1) begin to grow**, or **2) evaporate less quickly**. *[2.5 pts]*

[5 pts]
[20 pts total]

3. (a) The adiabatic LWC versus altitude under these specific conditions (when $e_\infty >> e_s(D_p)$, updraft velocity w is constant, droplet concentrations N_d have reached steady state and droplets grow only by condensation) can be derived as follows:

$$\frac{dLWC}{dz} = \frac{dLWC}{dD_p}\frac{dD_p}{dz} = \frac{d\left[N_d\rho_L(\pi/6)D_p^3\right]}{dD_p}\frac{dD_p}{dz} = \left[N_d\rho_L(\pi/2)D_p^2\right]\frac{dD_p}{dz}$$

$$= \left[N_d\rho_L(\pi/2)D_p^2\right]\left[\frac{1}{w}\frac{dD_p}{dt}\right] = \boldsymbol{\left[N_d\rho_L(\pi/2)D_p^2\right]\left[\frac{1}{w}\left(\frac{4D_v e_\infty}{D_p RTn_L}\right)\right]}$$

$$= \left[N_d\rho_L(\pi/2)D_p^2\right]\left[\frac{1}{w}\frac{4D_v e_\infty}{D_p R_v T\rho_L}\right] = \boldsymbol{\frac{2\pi N_d D_p D_v e_\infty}{wR_v T}}\, \boldsymbol{[5\ pts]}$$

[5 pts]

(b) A 60nm diameter haze droplet (thick black curve in the provided figure) takes **~300s (~5 min)** *[2.5 pts]* to grow to 10μm from the start of the simulation. To double in size to 20μm requires more than **1000s (~20 min)** *[2.5 pts]* of growth time. The slowing growth rate is due to the fact that diffusional growth is a function of $1/D_p$ and also because the ambient supersaturation is depleted by the condensational growth of the cloud droplet population. **Cloud droplets would not grow to 40 μm** by condensational growth alone *[2.5 pts]*. It would require that the cloud be **1200m** (~3600 ft) tall to sustain this level of droplet growth for 40 min, and the droplets would not even be close to 40μm by then (since a doubling in the growth time does not lead to a doubling in droplet size).
[7.5 pts]

(c) Notice that the conditions described in this problem are the same as in the cloud parcel model simulations shown in Fig.2 above) after about 1000 s (with 0.5m/s updraft velocity and LWC (or W_L) of 1 g/m$_3$). Curve (3) in the figure referenced (Fig.2b(right) from the homework) corresponds to an updraft velocity of 0.5m/s and a cloud top height of 1200m. In these simulations, which include the collision-coalescence process, and under these conditions, a "collector droplet" of 26μm can grow **as large as 1mm** *[2.5 pts]* by the time it reaches cloud base, becoming a raindrop. As discussed previously, cloud droplets would not even grow to 40μm under those conditions by condensation alone (which is true even if starting at 26μm, since the growth rate diminishes with droplet size).
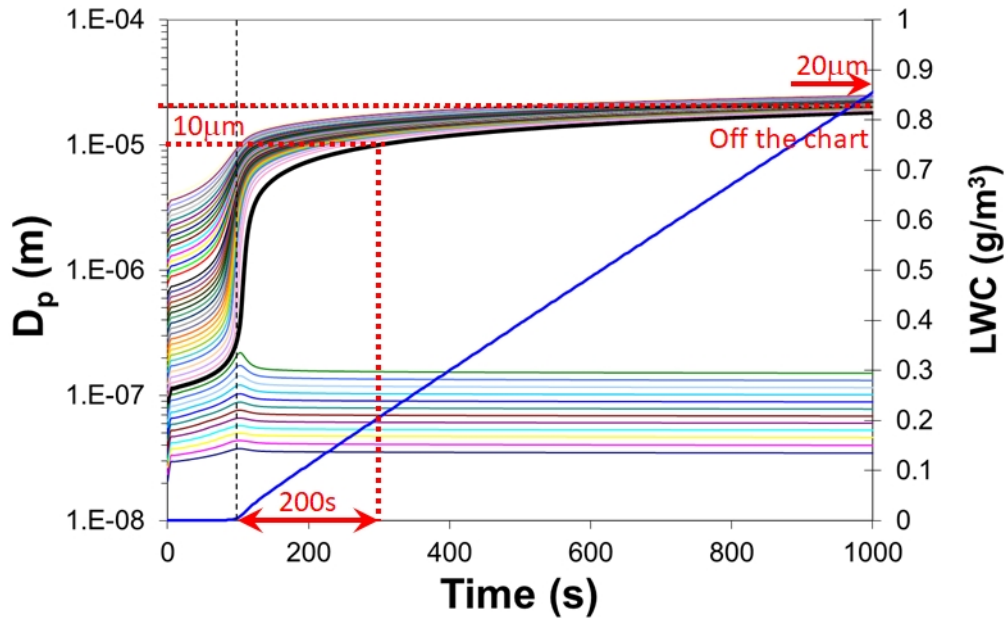*[2.5 pts]*

[15 pts total]

Figure 2: Cloud parcel simulation of droplet growth by condensation during constant uplift.

4. (a) *The following five points must be mentioned, with some level of equivalent description.*

   Point 1 – *disordered nature of liquid* [2 pts]: Liquid is made up of water molecules where some are bonded (via hydrogen bonds) and some are not, e.g., disordered bonding.

   Point 2 – *space created from less bonding* [2 pts]: This disordered nature allows more molecules to move freely and take up "space" that would otherwise be unavailable if forced to bond with other molecules.

   Point 3 – *rigid bonding* [2 pts]: In contrast, ice crystals contain rigid hydrogen bonds.

   Point 4 – *hexagonal structure* [2 pts]: The rigid bonds force the molecules into a hexagonal structure.

   Point 5 – *structure creates space* [2 pts]: The bond length required to maintain this structure creates more "space" in between the molecules,

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reducing the density from 1000 to 920 for ice. The 80 kg m^{-3} accounts for the space created when the hydrogen bonds form.

Note: half credit if explanation missing but includes molecular drawing.

[10 pts]

(b) *The following three points must be mentioned, with some level of equivalent description.*

Point 1 – *density decreases* [4 pts]: The effective density decreases below 920 kg m^{-3} when $\phi \neq 1$ (non-spherical crystals).

Point 2 – *gradients at largest curvature increase growth* [4 pts]: During diffusional growth, the vapor density gradients are concentrated at the edges/tips of the particle (where curvature is the greatest). This forces the vapor to deposit at these edges, rather than directly onto the face.

Point 3 – *hollowing* [2 pts]: As this continues, a hollowing occurs within these crystals, reducing the particle density.

Note: drawings including gradients & flux may supplement point 2 above.

[10 pts]

[10+10=20 pts]

5. (a) (i)

- *[2 pts]* For a plate:

$$\frac{dm}{dt} = 4\pi a f_{ob}(\phi) G_i S_i \tag{1}$$

- For a spheroid:

$$m = \frac{4}{3}\pi a^2 c \rho_i \tag{2}$$

- *[2 pts]* Take the derivative of (2) with respect to a only:

$$\frac{dm}{dt} = \frac{4}{3}\pi (2a) c \rho_i \frac{da}{dt} \tag{3}$$

- *[1 pt]* Set (1) equal to (3):

$$4\pi a f_{ob}(\phi) G_i S_i = \frac{8}{3}\pi a c \rho_i \frac{da}{dt} \tag{4}$$

- **[1 pt]** Solve for $\frac{da}{dt}$:

$$\frac{da}{dt} = \frac{3f_{ob}(\phi)G_iS_i}{2c\rho_i} \tag{5}$$

- **[2 pts]** Solve for $a(t)$ by integrating from $t = 0 \to t$ and know that $r_o = a_o$:

$$\int_{a_o}^{a(t)} da = \frac{3f_{ob}(\phi)G_iS_i}{2c\rho_i} \int_0^t dt \tag{6}$$

$$a(t) = a_o + \frac{3f_{ob}(\phi)G_iS_i}{2c\rho_i}t \tag{7}$$

(ii)

- **[2 pts]** For a column:

$$\frac{dm}{dt} = 4\pi c f_{pr}(\phi)G_iS_i \tag{8}$$

- For a spheroid:

$$m = \frac{4}{3}\pi a^2 c\rho_i \tag{9}$$

- **[2 pts]** Take the derivative of (9) with respect to c only:

$$\frac{dm}{dt} = \frac{4}{3}\pi a^2 \rho_i \frac{dc}{dt} \tag{10}$$

- **[1 pt]** Set (8) equal to (10):

$$4\pi c f_{pr}(\phi)G_iS_i = \frac{4}{3}\pi a^2 \rho_i \frac{dc}{dt} \tag{11}$$

- **[1 pt]** Solve for $\frac{dc}{dt}$:

$$\frac{1}{c}\frac{dc}{dt} = \frac{3f_{pr}(\phi)G_iS_i}{a^2\rho_i} \tag{12}$$

- **[2 pts]** Solve for $c(t)$ by integrating from $t = 0 \to t$ and know that $r_o = c_o$:

$$\int_{c_o}^{c(t)} \frac{dc}{c} = \frac{3f_{pr}(\phi)G_iS_i}{a^2\rho_i} \int_0^t dt \tag{13}$$

$$\ln\left[\frac{c(t)}{c_o}\right] = \frac{3f_{pr}(\phi)G_iS_i}{a^2\rho_i}t \tag{14}$$

$$c(t) = c_o \exp\left[\frac{3f_{pr}(\phi)G_iS_i}{a^2\rho_i}t\right] \tag{15}$$

[1 pt] Physical Dependencies:
$a(t) \rightarrow$ linear increase in time
$c(t) \rightarrow$ exponential increase in time

(b) Because the temperature is -15°C, we assume that the crystal is a plate, so the major axis length is the a axis. Thus, we solve for the change in a in time and assume all else is constant *[2 pts]*.

[2 pts] $a(t) = a_o + \frac{3f_{ob}(\phi)G_iS_i}{2c\rho_i}t$
[2 pts] $a(t) = 25\mu m + \frac{3(0.6)(1\times10^{-8}\ \text{kg m}^{-1}\ s^{-1})(0.15)}{2(25\times10^{-6}\ \text{m})(920\ \text{kg m}^{-3})}\left(15\min \cdot \frac{60\ s}{\min}\right)$
[2 pts] $a(t) = 77.8\mu m$