Interpretations of the angular-momentum principle as applied to the general circulation of the atmosphere

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In two recent articles dealing with the general circulation each of the writers of this note had occasion to discuss the concept of angular momentum about the earth's axis. According to the conclusions reached by Rossby[1] the principle of the conservation of angular momentum is found to be unsuited for elucidating global atmospheric circulations, while according to the conclusions reached by Starr[2] this principle is a valuable tool in studying large-scale air motions. We wish to point out that the seeming disparity of views is superficial, and that in fact we are in fundamental agreement concerning the particular interpretations involved in each of the two discussions as far as this topic is involved.

The equation of motion for the zonal direction may be written in the form

$$\rho \frac{dM}{dt} = \frac{\partial \rho M}{\partial t} + \nabla \cdot \rho M \boldsymbol{c} = - r \frac{\partial p}{\partial x} - T,$$

in which M is absolute angular momentum per unit mass, ρ the density, $\boldsymbol{c}$ the absolute vector velocity, r the distance from the earth's axis, p pressure, T the frictional retarding torque per unit mass, x linear distance eastward, and t time. The equation states that the rate of change of absolute angular momentum is equal to the sum of the torques due to pressure and to friction. The first equality is obtained with the aid of the continuity equation $\partial\rho/\partial t + \nabla \cdot \rho \boldsymbol{c} = 0$.

It is possible to form a volume integral of the equation given above over an annular zonal strip in the atmosphere bounded by vertical surfaces at two arbitrary latitudes, by the earth's surface (assumed to be uniform), and by some arbitrary level a (constant) vertical distance above the surface. Thus, with the aid of the divergence theorem,

$$\frac{\partial}{\partial t} \int \rho M \, dV = \int \rho M c_n \, dS - \int T \, dV,$$

where dV is a volume element, dS an element of the bounding surface, and c_n the inward component of $\boldsymbol{c}$ at the boundary. In this integration the contribution from the pressure term vanishes because of cyclic continuity of p. We also have the relation that $M = rU = ru + r^2\omega$, U being the eastward component of $\boldsymbol{c}$, ω the earth's angular velocity, and u the eastward relative velocity. The preceding equation may therefore be rewritten in the form

$$\frac{\partial}{\partial t} \int r\rho u \, dV = - \frac{\partial}{\partial t} \int \rho r^2 \omega \, dV + \int r\rho u c_n \, dS + \int r^2 \omega \rho c_n \, dS - \int T \, dV.$$

This equation states that within this space the relative angular momentum (represented by the integral in the term on the left-hand side) may change in consequence of

a. A change of the angular momentum due to the earth's rotation, represented by the first term on the right-hand side. This effect depends upon progressive redistribution of mass within the annular space, and is on the average zero.

b. A flux of relative angular momentum into the space across the boundary due to advection, represented by the second term on the right-hand side.

c. A flux of angular momentum due to the earth's rotation across the boundary, represented by the third term on the right-hand side.

d. A retardation due to friction, represented by the last term on the right.

Since in the long run the changes of relative angular momentum are zero, it follows that the frictional retardation is balanced by the contributions listed under (*b*) and (*c*). In order to secure a contribution from (*c*) it is necessary that

$$\int \rho c_n \, dx \neq 0,$$

where the integration in the zonal direction extends along the entire circumference of a latitude circle. In other words it is required that there exist *mean meridional circulations* in the atmosphere. *Most of the classic theories for the general circulation were based upon the assumption that it is this effect of meridional circulations which maintains the angular momentum of the zonal motions in the atmosphere.* It is this assumption that *both of us* call into question for reasons enumerated by Rossby.[1]

Following an earlier analysis by Jeffreys the study of the term noted under (*b*) was pursued further by Starr,[2] this being the only alternative available. The only questions which can arise in this connection concern the nature and scale of the disturbances which are important and how their effects might be measured from data. However, we concur in the opinion that it is this term—the advective transport of relative angular momentum—which is of prime importance in the mechanics of the general circulation.

[1] C.-G. Rossby, "On the nature of the general circulation of the lower atmosphere," chapter 2 in *The atmospheres of the earth and planets*, G. P. Kuiper, editor. Univ. Chicago Press, 1949.

[2] Victor P. Starr, "An essay on the general circulation of the earth's atmosphere," *J. Meteor*, **5**, 39–43, 1948.