ON THE PRODUCTION OF KINETIC ENERGY IN THE ATMOSPHERE

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ABSTRACT

In the present paper the production of kinetic energy in the atmosphere is examined from a hydrodynamical point of view. The results indicate that the intensity of the primary source of horizontal kinetic energy at any point in the atmosphere is equal to the pressure multiplied by the horizontal divergence of the velocity. Regions of horizontal velocity convergence appear as hydrodynamical sinks for kinetic energy, in addition to frictional effects. It is found that kinetic energy may be transferred through advection and through work done by pressure forces. It appears that diverging anticyclones are of primary importance in providing kinetic energy for the general circulation.

1. Introduction

One of the basic problems in the science of meteorology relates to the manner in which thermal energy received by the atmosphere through short-wave solar radiation becomes in part transformed into kinetic energy of motion relative to the rotating earth. Plausible estimates show that the fraction of the total energy so transformed is very small, but must nevertheless be sufficient to account for all air motions, in the absence of any other significant energy sources. Since the kinetic energy of organized motions is continually degraded and ultimately dissipated by turbulence and viscosity, the process of kinetic energy production must be a continuous one with, probably, certain fluctuations about a mean rate when the whole atmosphere is considered. The purpose of this paper is to examine this production process from a hydrodynamical point of view.

Changes in the kinetic energy of a particle or system of particles can result only from the action of mechanical forces, and hence the rate of kinetic-energy production can be discussed in terms of the joint action of such forces and the kinematics of existing motions. In this light it is not essential to inquire how systems of such forces and such motions in the atmosphere are related to the thermodynamical processes which are ultimately responsible for their existence. In order to demonstrate the particular point in question as simply as possible we shall first consider an example of fluid motion under somewhat artificial circumstances, but still having theoretical interest. In view of the fact that the kinetic energy of vertical motions in the atmosphere is very small compared with the kinetic energy of the large-scale horizontal motions, we shall consider only the latter.

The approach used is one suggested by the beautiful classic paper of Osborne Reynolds (1895) entitled "On the dynamical theory of incompressible viscous fluids and the determination of the criterion." Whereas Reynolds was concerned only with the dissipation of kinetic energy, his treatment must be modified in order to envisage also the process which creates kinetic energy. For this reason his assumption of incompressibility will be abandoned. Also, our restriction to the study of the kinetic energy of horizontal motions introduces certain changes, although these changes are not actually in the nature of approximations.

2. Study of a simple system

Let it be supposed that a mass of gas is confined in a chamber with a plane bottom and vertical walls, under the action of gravity which we assume to be acting vertically downward. If the chamber is of sufficiently great height, it is not necessary that it have a top. Likewise, the gas need not be an ideal one, since no use will be made of an equation of state. Coriolis forces will, for the present, be omitted. Let it be supposed further that the gas is in some state of motion induced by differential heating and cooling.

If we take x, y, z to be a cartesian coordinate system with the positive z-axis vertical, we may write the equations of motion for the horizontal directions in the form

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x, \qquad \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y. \tag{1}$$

Here u, v are the velocity components in the directions x, y; ρ is the density; p the pressure; t time; and F_x, F_y are the components of the viscous forces in the x, y directions. Generally speaking, the motions in the chamber might be turbulent. If we wish to regard the dependent variables in equations (1) as representing mean values free of the turbulence components, we shall assume that the only change necessary is to include eddy-stress effects in the quantities F_x, F_y after the manner of Reynolds. More will be said concerning this point later.

The kinetic-energy equation corresponding to the system (1) is

$$\rho\frac{\partial}{\partial t}\frac{V_h^2}{2}+\rho u\frac{\partial}{\partial x}\frac{V_h^2}{2}+\rho v\frac{\partial}{\partial y}\frac{V_h^2}{2}+\rho w\frac{\partial}{\partial z}\frac{V_h^2}{2}=-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)-d. \quad (2)$$

We use the symbol d to represent the rate at which the turbulence and viscosity are decreasing the kinetic energy per unit volume, and $V_h^2 \equiv u^2+v^2$. It is possible to rewrite (2) in the following form:

$$\frac{\partial E}{\partial t}+\frac{\partial Eu}{\partial x}+\frac{\partial Ev}{\partial y}+\frac{\partial Ew}{\partial z}=-\left(\frac{\partial pu}{\partial x}+\frac{\partial pv}{\partial y}\right)+p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-d, \quad (3)$$

where use has been made of the continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}=0, \quad (4)$$

which in any case must be true, and where $E \equiv \frac{1}{2}\rho V_h^2$ is the horizontal kinetic energy per unit volume. The quantity represented by the last three terms on the left-hand side of (3) is the divergence of the (three-dimensional) kinetic energy transport vector EV. The quantity in the first parentheses on the right is the divergence of the horizontal vector pV_h. If equation (3) is integrated over an arbitrary volume, both of these quantities may be represented as surface integrals with the aid of the divergence theorem. Thus, if the limits are fixed, we may write

$$\frac{\partial}{\partial t}\iiint E\,dx\,dy\,dz = \int EV_n\,dS - \iint p(v\,dx-u\,dy)dz + \iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dx\,dy\,dz - \iiint d\,dx\,dy\,dz, \quad (5)$$

where V_n is taken to be the inward component of velocity at the boundary, and dS is a surface element.

Equation (5) may now be given the following interpretation. The total horizontal kinetic energy (T) in a fixed region may be changing in consequence of:

a. An advection of new fluid having kinetic energy across the boundary. This is represented by the term

$$A=\int EV_n\,dS.$$

This is then one mode of *redistribution* of kinetic energy.

b. The performance of work by pressure forces at the boundary in virtue of the displacements due to the horizontal velocity components. This is represented by the term

$$W=-\iint p(v\,dx-u\,dy)dz.$$

This is a second mode of *redistribution* of kinetic energy.

c. A production of kinetic energy within the volume itself. This is represented by the term

$$S=\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dx\,dy\,dz,$$

which contains the *primary source* of kinetic energy.

d. The action of frictional forces. This effect would ordinarily consist of a dissipation and is represented by the term

$$D=\iiint d\,dx\,dy\,dz.$$

If the limits of integration include all of the fluid in the fixed chamber, it is clear that the surface integrals must vanish, so that in a *mechanically closed* system (5) reduces to

$$\frac{\partial T}{\partial t}=S-D. \quad (6)$$

Since for such a system the frictional effect would ordinarily lead to dissipation, it follows that S must be positive if the kinetic energy T is to remain constant or increase. If a more or less constant amount of kinetic energy is to be present, the dissipation must be balanced by a corresponding positive average rate of production.

The production S may be looked upon as the integral of the contributions from the various horizontal layers of fluid present and written as

$$S=\int\left[\iint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dx\,dy\right]dz. \quad (7)$$

In view of the fact that the surface integral

$$\iint\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dx\,dy$$

must vanish if the horizontal velocity is zero across the fixed walls, it follows that a given horizontal stratum of fluid cannot give a positive contribution to S unless larger values of the pressure p are associated with areas of horizontal divergence than are associated with areas of convergence. Thus areas of *horizontal* divergence represent primary kinetic energy sources while areas of convergence represent sinks for kinetic energy. Furthermore, in a mechanically closed system of the kind here considered it is impossible to have source regions for kinetic energy without at the same time having sinks of a hydrodynamic nature, entirely independent of frictional effects.

3. Equations for the atmosphere

Before embarking upon a discussion of the meteorological implications of the material presented above, it is desirable to develop the concepts involved in more general terms, so as to render it possible to perform integrations over the entire mass of the atmosphere.

To a sufficiently close degree of approximation the shape of the geopotential surfaces may be considered as spherical so that we may make use of spherical polar coordinates in which r is the radius, ϕ is latitude, and λ is longitude. By analogy with the cartesian case we may then write the equations of motion for the horizontal directions (see Brunt, 1939) in the form

$$\left.\begin{aligned} \frac{du}{dt} - \frac{uv}{r}\tan\phi + \frac{uw}{r} + 2\Omega(w\cos\phi - v\sin\phi) & \\ = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x, & \\ \frac{dv}{dt} + \frac{u^2}{r}\tan\phi + \frac{vw}{r} + 2\Omega u\sin\phi & \\ = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_y, & \end{aligned}\right\} \quad (8)$$

where u, v, w are the linear velocity components in the northward, eastward, and upward directions, respectively, and x, y are measures of linear distance eastward and northward, respectively; Ω is the angular velocity of the earth. The analogous energy equation in this case may be written as

$$\begin{aligned} \rho\frac{d}{dt}\frac{V_h^2}{2} + \rho\frac{V_h^2}{r}w + 2\rho\Omega uw\cos\phi & \\ = -\left(\frac{\partial pu}{\partial x} + \frac{\partial pv}{\partial y} - \frac{pv}{r}\tan\phi\right) & \\ + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{v}{r}\tan\phi\right) - d. & \quad (9) \end{aligned}$$

Making use of the observational fact that the last two terms on the left-hand side of (9) are of a very small order of magnitude, these terms will be dropped.[1]

[1] In reality these terms represent a conversion of kinetic energy of horizontal motions into kinetic energy of vertical motions, and as such do not involve a production of kinetic energy. Indeed, by methods similar to those used in this paper one can investigate separately the kinetic energy of motions in each of the three directions, namely, zonal, meridional and vertical. In that case other conversion terms of a similar nature arise.

The manipulation of the remaining term on the left side may now be carried out with the aid of the continuity equation much as before, since this operation is independent of the specific coordinate system used, so that we may write

$$\frac{\partial E}{\partial t} + \operatorname{div}_3 EV = -\operatorname{div}_2 pV_h + p\operatorname{div}_2 V_h - d. \quad (10)$$

A volume integral of (10) may now be taken and written in the form

$$\frac{\partial}{\partial t}\int E\,d\tau = \int EV_n\,dS - \int\int p(v\,dx - u\,dy)dr + \int p\operatorname{div}_2 V_h\,d\tau - \int d\,d\tau, \quad (11)$$

where $d\tau$ is a volume and dS a surface element. Equation (11) is physically identical with (5) and has, therefore, the same interpretation. In symbolic form we may write

$$\frac{\partial T}{\partial t} = A + W + S - D, \quad (12)$$

which states that the rate of increase of horizontal kinetic energy for a fixed volume is equal to the net rate of advection of such kinetic energy into the region, plus the rate at which work is being done by the surroundings on the fluid in the region through horizontal motions, plus the production of kinetic energy in the volume, minus the frictional dissipation. For a system which is mechanically closed A and W again vanish. This is therefore true when the entire atmosphere is considered. In this case the surface integral of the horizontal divergence over each closed geopotential surface must vanish as in the case of the chamber previously considered.

4. Conclusions

Although it is possible to form other energy integrals for fluid motion, as pointed out in standard texts on hydrodynamics,[2] the particular merit of the procedure followed above is that the expression for production of kinetic energy assumes a form which is of interest in meteorological problems. The implications of equation (12) may be stated in brief as follows:

a. The intensity of the primary source of horizontal kinetic energy at a given point in the atmosphere is given by the product of the pressure into the divergence of the horizontal velocity.

b. Positive primary sources must always occur in combination with negative sources or sinks independent of frictional effects, when the entire atmosphere is considered.

[2] See, for example, Bjerknes *et al.* (1933).

c. In addition to the action of the sources and frictional effects, the horizontal kinetic energy in a fixed region not embracing the entire atmosphere may change due to advection of kinetic energy across the boundary and due to the redistribution of kinetic energy through the boundary by work done by pressure forces and horizontal velocity components at the boundary.

From the standpoint of the general circulation it would appear that the sources of kinetic energy are to be found in the regions of horizontal divergence. The net contribution from a given level results from the fact that areas of divergence generally occur at a different pressure than do the areas of convergence. Thus at lower levels it is common for horizontal divergence to be present in anticyclonic areas while convergence takes place in cyclonic areas, the net result being positive. We as yet do not have sufficient observational material concerning the distribution of divergence at higher levels, but the fact that the pressure decreases with elevation would seem to indicate that the importance of the higher levels rapidly diminishes. Generally speaking, it would thus appear that the energy sources for the general circulation are to be found principally in the subtropical high-pressure cells, the migratory polar anticyclones and the subsiding cap of cold air over the polar regions. From these primary centers the kinetic energy is continually transferred to the cyclonic areas with convergence which act as sinks in addition to the action of friction.

One might ask why it is that if the diverging anticyclones act as primary sources of kinetic energy, they are not the scenes of major activity. Actually, however, the generation process cannot be present in such systems without the simultaneous operation of the transfer processes. If divergence exists in an anticyclone, the peripheral outward motion results in a rapid outward flow of kinetic energy through work done by pressure forces and through advection.

We have made the tacit assumption in the development given above that the "frictional" term D leads to a dissipation of kinetic energy. If only molecular viscosity and small-scale turbulent viscosity are included in this term the assumption is undoubtedly valid. However, if relatively large-scale eddies and other large features of the atmospheric motions are included in the form of a gross turbulence as distinguished from the remaining mean motion, it is apparent that the quantity D may then embrace energy-producing systems and it is possible that it may change sign. Thus, for example, if only the average zonal circulation of the atmosphere be considered as the true mean motion so that the cyclones, anticyclones and other nonzonal motions appear as turbulence, there is no clear *a priori* reason for assuming that the term D represents a dissipation.

Finally, it is interesting to compare the results obtained here with those of Margules (1905) in his classic paper, "On the energy of storms." Very broadly speaking the two approaches deal with essentially the same process. We have simply enlarged the "chamber" containing the gas used by Margules so as to include the whole atmosphere. Furthermore, whereas Margules considered a discrete process, we have replaced it by a continuous one and restricted our attention to the production, redistribution, and dissipation of kinetic energy of horizontal motions only. Also, we have recognized that under these circumstances the pressure multiplied by the *horizontal* divergence is the measure of the rate at which other forms of energy such as potential and internal energy are being converted into kinetic energy.[3] When the divergence is negative the sense of this conversion process is reversed. It should be noted that this result is independent of the physical nature of the "working substance," which might indeed be partly liquid (or even solid), with the gaseous and liquid components undergoing changes of phase. The result therefore automatically embraces the consequence of all condensation phenomena insofar as they contribute to the horizontal kinetic energy.

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[3] It is worthy of note that the present treatment gives no information as to whether the bulk of the kinetic energy generated in the atmosphere represents a conversion from geopotential energy or whether it represents a conversion directly from internal heat energy.