ATM 316: Dynamic Meteorology I
Final Review, December 2014

Scalars and Vectors
- Scalar: magnitude, without reference to coordinate system
- Vector: magnitude + direction, with reference to coordinate system

Basic Vector Operations
- Vector addition / subtraction
- Scalar multiplication
- Dot product $\vec{A} \cdot \vec{B}$
    - Input: 2 vectors. Output: scalar
    - Projects $\vec{A}$ onto $\vec{B}$ ("shadows") and then multiplies lengths together
    - $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\,\cos\theta$
    - Dot product of perpendicular vectors is always zero
- Cross Product $\vec{A} \times \vec{B}$
    - Input: 2 vectors. Output: vector
    - Magnitude is area of parallelogram spanned by $\vec{A}$ and $\vec{B}$, $\vec{A} \times \vec{B} = |\vec{A}|\,|\vec{B}|\,\sin\theta$
    - Direction is perpendicular to both $\vec{A}$ and $\vec{B}$. Use right hand rule.
    - Cross product of parallel vectors is always zero.

Vector derivatives
- **Gradient**
    - Input: scalar function, $f(x, y, z)$
    - Output: vector function, $\nabla f = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$
    - Points in direction of steepest increase of f
    - Magnitude determined by the rate of change of f over distance.
- **Divergence**
    - Input: vector function, $\vec{b}(x, y, z) = f(x, y, z)\hat{\imath} + g(x, y, z)\hat{\jmath} + h(x, y, z)\hat{k}$
    - Output: scalar function, $\nabla \cdot \vec{b} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$
    - Measures whether something is accumulating or evacuating from each point.
- **Curl**
    - Input: vector function, $\vec{b}(x, y, z) = f(x, y, z)\hat{\imath} + g(x, y, z)\hat{\jmath} + h(x, y, z)\hat{k}$
    - Output: vector function, $\nabla \times \vec{b} = \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right)\hat{\imath} + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\hat{k}$
    - Measures the spin/rotation of a vector field. Direction is determined by right hand rule
    - The curl of the velocity field is called the **vorticity**
- **Laplacian**
    - Input: scalar function, $f(x, y, z)$
    - Ouput: scalar function, $\nabla^2 = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
    - Applications to diffusion of heat and substances.

Taylor Series approximation

$$f(x) = f(x_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial^2 f}{\partial x^2}\frac{(x - x_0)^2}{2!} + \cdots + \frac{\partial^n f}{\partial x^n}\frac{(x - x_0)^n}{n!} + \cdots$$

- For very small distances from a reference point, higher order terms are negligible:

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}(x - x_0)$$

Kinematics of horizontal flow

- **Divergence** of the horizontal wind: $\delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$
- **Vorticity** of the horiztonal wind: $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ (the vertical component of the vorticity vector)
- **Stretching deformation**: $d_1 = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}$
- **Shearing deformation**: $d_2 = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$
- **Total deformation**: $d = \sqrt{{d_1}^2 + {d_2}^2}$
- δ, ζ, d are all **rotationally invariant** (i.e. don't depend on the orientation of the coordinate system).

Newton's Second Law of motion

- $\sum_i \vec{\boldsymbol{F}_\iota} = m\,\vec{\boldsymbol{a}}$
- The physical law that dictates the motion of fluid parcels, whence we derive the "momentum equations" or "equations of motion" for a fluid.

Fundamental forces in a fluid

- **All forces express per unit mass (i.e. in units of acceleration)**
- **Pressure gradient force**
  - Arises whenever there are spatial variations in pressure in the fluid
  - Causes acceleration toward lower pressure: $\vec{F}_{PGF} = -\frac{1}{\rho}\nabla p$
- **Gravitational force**
  - Arises between any two bodies with mass
  - In this case, the air parcel and the solid earth.
  - $\vec{F}_{grav} = -\frac{G\,M}{|\vec{r}|^2}\hat{r} \approx -g\,\hat{k}$, where $g = 9.8 \text{ m s}^{-2}$
- **Frictional force**
  - Arises when there is resistance to flow, particularly near solid surfaces
  - Tends to oppose the motion.

Eulerian versus Lagrangian derivatives

- Both are time derivatives or tendencies (rate of change with time) of a fluid property.
- The **Eulerian** derivative $\frac{\partial}{\partial t}$ is the tendency at **fixed points** (e.g. as measured by stations)
- The **Lagrangian** derivative $\frac{D}{Dt}$ is the tendency for **fixed air parcels** (following the motion, e.g. as measured by a balloon that is drifting with the flow).
- Defined for any scalar fluid quantity $f(x, y, z)$ (e.g. density, pressure, temperature as

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f$$

- Or in component form, $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$
- Also called "material derivative" or "total derivative".

- Lagrangian derivative of a vector field is a vector whose components are just the Lagrangian derivative of each component:

$$\frac{D\vec{b}}{Dt} = \left(\frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f\right)\hat{\imath} + \left(\frac{\partial g}{\partial t} + \vec{u} \cdot \nabla g\right)\hat{\jmath} + \left(\frac{\partial h}{\partial t} + \vec{u} \cdot \nabla h\right)\hat{k}$$

Advection

- Time tendency of a fluid property (e.g. temperature) at fixed points due to fluid motion.
- E.g. warm air advection occurs at a point if the wind at that point is coming from a location with warmer temperature.
- Advection of scalar f is calculated by setting $\frac{Df}{Dt} = 0$ (no tendency for fixed parcels), so

$$\frac{\partial f}{\partial t} = -\vec{u} \cdot \nabla f$$

Momentum equation for a frictionless, non-rotating fluid

- Derived by applying Newton's second law to fluid parcels: $\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p - g\,\hat{k}$
- Or in component form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{x}}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{y}}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial \mathrm{p}}{\partial \mathrm{z}} - g$$

Hydrostatic balance

- Approximate balance between gravity and the vertical component of the PGF
- Derived by setting $\frac{Dw}{Dt} = 0$ in the vertical component of the momentum equation

$$\frac{\partial \mathrm{p}}{\partial \mathrm{z}} = -\rho g$$

- Density related to temperature and pressure through the ideal gas law, $p = \rho\, R_d\, T$
- Combining ideal gas law and hydrostatic balance and integrating vertically gives the hypsometric equation: $\Delta z = H \ln\left(\frac{p_1}{p_2}\right)$
- The scale height is determined by column-average temperature: $H = \frac{R_d\,\bar{T}}{g}$
- Hypsometric eq. says that warmer columns are thicker than colder columns.
- Pressure decreases approximately exponentially in the vertical: $p(z) = p(0)\,\exp\left(-\frac{z}{H}\right)$

Frames of reference

- **Inertial frame**: a coordinate system that is **not accelerating** with respect to fixed space
- **Non-inertial frame**: a coordinate system that **is accelerating** with respect to fixed space
- To apply Newton's law in a non-inertial frame, we need to introduce **apparent forces** to account for the acceleration of the coordinate system in which we measure the motion.

Rotation and rotating frames of reference

- Measure rotation with rotation rate Ω in radians per second, or a rotational period $T = \frac{2\pi}{\Omega}$
- Azimuthal velocity of a particle or fluid parcel traveling in a circular path: $v_\theta = \Omega r$

- Any body or fluid parcel that travels in a circular path is accelerating toward the center of the circle. We call this the **centripetal acceleration**, $\vec{a} = -\frac{v_\theta{}^2}{r}\hat{r}$. By Newton's law, there must be a net force radially inward supplying this acceleration (as measured in the **inertial frame**).
- Alternatively, if we are measuring the motion from the **rotating frame of reference**, there is a **centrifugal force** acting radially outward on the particle. Its magnitude is equal and opposite to the centripetal acceleration.
- The free surface of a shallow water tank in solid body rotation

$$h(r) - h(0) = \frac{\Omega^2 r^2}{2g}$$

- Two ways to describe this force balance:
    - Inertial frame: a component of gravity acts down the slope, which is radially inward. This is the only force. The net force is thus radially inward. This provides the centripetal acceleration needed to keep water parcels moving in circular paths.
    - Rotating frame: a component of gravity acts down the slope, radially inward. It is balanced by a centrifugal force that acts radially outward. The net force is zero. Therefore the fluid parcels are stationary in the rotating frame.
- Momentum equations in the rotating frame, including the **apparent forces**

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi - 2\vec{\Omega}\times\vec{u}$$

- $\vec{\Omega}$ is the **rotation vector** of the Earth, it points up through the north pole.
- $\left|\vec{\Omega}\right| = \frac{2\pi}{\text{day}}$ (we often just write Ω for the magnitude).
- Definition of **geopotential**:

$$\Phi = gz - \frac{\Omega^2 r^2}{2}, \qquad r = a\cos\phi$$

- The **gradient of the geopotential** is the vector sum of the gravitational force (toward center of Earth) plus the centrifugal force associated with the rotation of the Earth (radially outward from the axis of rotation)
- Rotating planets (including Earth) are **oblate spheroids** – wider at the equator than at the poles – due to centrifugal force. The surface of the planet is approximately a surface of constant geopotential.
- For most problems we ignore the very small correction due to centrifugal force and just define the **geopotential height** as $Z = \Phi/\mathrm{g}$
- The Coriolis force is $-2\vec{\Omega}\times\vec{u}$, always acts at perpendicular to the flow $\vec{u}$.

Simplified momentum equations for mid-latitude synoptic-scale flow:

- Scale analysis for mid-latitude synoptic-scale flow shows that
    - Vertical accelerations are negligible
    - Non-horizontal components of the Coriolis force are negligible.
    - Horizontal accelerations are an order of magnitude smaller than PGF and Coriolis force, but not completely neglible.
    - Simplified equations are

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \mathrm{x}} + fv, \qquad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial \mathrm{y}} - fu, \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial \mathrm{z}} - \frac{\partial \Phi}{\partial z}$$

    - Definition of Coriolis parameter: $f = 2\Omega\sin\phi$, where ϕ is latitude.

- Vertical component is just hydrostatic balance.
- Coriolis force always accelerates a parcel to its right in the NH (to left in SH).

Geostrophic balance

- If the horizontal accelerations are zero, there is a balance between PGF and Coriolis force
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial \mathrm{x}} + fv, \qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial \mathrm{y}} - fu$$
- Use these balances to define the geostrophic wind vector $\overrightarrow{v_g} = u_g\hat{\imath} + v_g\hat{\jmath}$ as follows:
$$\overrightarrow{v_g} = \frac{1}{\rho f}\hat{k}\times\nabla p$$
- Or in component form,
$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial \mathrm{y}}, \qquad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial \mathrm{x}}$$
- $\overrightarrow{v_g}$ always points along the isobars, with high pressure to the right (left) in the NH (SH)
- To calculate $\overrightarrow{v_g}$, need to know the pressure field, the latitude (f), and density. Not the wind field!

Isobaric (pressure) coordinates

- Pressure is a very useful vertical coordinate because
  - Pressure decreases monotonically with height (hydrostatic balance)
  - Vertical pressure variations >> horizontal pressure variation
- Pressure gradient force (horizontal component) in isobaric coordinates acts towards low geopotential height on pressure surfaces. Almost identical to low pressure on height surfaces.
- Horizontal equations of motion in pressure coordinates (in the rotating frame):
$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial \mathrm{x}} + fv, \qquad \frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial \mathrm{y}} - fu$$
- The derivatives are understood to be taken **at constant pressure**.
- Can also be written in vector form, with $\vec{v} = u\hat{\imath} + v\hat{\jmath}$ the horizontal wind vector:
$$\frac{D\vec{v}}{Dt} = -\nabla_p\Phi - f\hat{k}\times\vec{v}$$
- Vertical motion in pressure coordinates is $\omega = \frac{Dp}{Dt}$ (time rate of change of pressure of air parcels), and $\omega < 0$ for upward motion.
- Geostrophic wind in pressure coordinates is $\overrightarrow{v_g} = \frac{1}{f}\hat{k}\times\nabla_p\Phi$, or in component form
$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial \mathrm{y}}, \qquad v_g = \frac{1}{f}\frac{\partial \Phi}{\partial \mathrm{x}}$$
- In isobaric coordinates the density no longer appears in the equations of motion.
- Geostrophic wind is determined by the geopotential field (and latitude).
- Geostrophic wind is parallel to contours of constant geopotential height, with high heights to the right (left) in the NH (SH).

Continuity Equation (conservation of mass)

- The mass of a fluid parcel is constant following the motion. This lets us write a relationship between density and the velocity field.
- In **height coordinates**, we derived two equivalent statements of mass conservation:
  - **Eulerian form**: $\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho\vec{u})$

- **Lagrangian form**: $\frac{D\rho}{Dt} = -\rho(\nabla \cdot \vec{u})$

- For an **incompressible fluid**, the density of parcel is constant and we can write $\nabla \cdot \vec{u} = 0$
- There is thus a **relationship between vertical motion and horizontal convergence / divergence**: $\frac{\partial w}{\partial z} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$
- Since $w = 0$ at a solid boundary (Earth's surface), we can infer that **vertical motion must be upward** above a region of **horizontal convergence** near the surface.
- In **pressure coordinates**, the full continuity equation is simply $\frac{\partial \omega}{\partial p} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ (with derivatives now taken at constant pressure).
- Thus the statement above linking horizontal convergence at the surface to upward motion is true even for a **fully compressible fluid** like the atmosphere.

Rossby Number

- A non-dimensional number defined as the **ratio** of **inertial acceleration** (advection of the velocity field) to acceleration due to **Coriolis force**.
- In practice we use $Ro = \frac{U}{f\,L}$
- **Small Rossby number** ($Ro \ll 1$) is associated with **geostrophically balanced** flow.
- On synoptic maps, Rossby number is ~ 0.1 (small) in many region, but approaches 1 in regions of **strong curvature**.
- We can think of the Rossby number as a ratio of timescales: Ro = (timescale of Earth rotation) / (advective timescale of motion). Motions that are fast compared to Earth's rotation (i.e. much faster than a day) have large Rossby number. This means that Coriolis force is probably not important for that motion.

Natural Coordinates

- We can rotate our usual horizontal x,y coordinates to **align with the direction of the flow** at each point.
- $\hat{t}$ is parallel to (and in the same direction as) the velocity vector at every point.
- $\hat{n}$ is perpendicular to the velocity vector at every point, and positive to the left.
- The horizontal velocity vector is therefore $\vec{v} = +V\hat{t}$, where V is the wind speed.
- We transform the horizontal momentum equations into $\hat{t}$ and $\hat{n}$ components. In the $\hat{t}$ direction (the along-flow component) we get

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s}$$

- This says that the parcel velocity can only change if the flow is crossing contours of constant geopotential height.
- In the $\hat{n}$ direction (cross-flow component) we get

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

- R is the radius of curvature following the air parcel. We define $R > 0$ for counter-clockwise curvature, $R < 0$ for clockwise curvature (same in both hemispheres).
- The three terms are: centrifugal force, Coriolis force, and PGF.

Balanced flow

- A balanced flow is a flow in which the wind field is in force balance with the geopotential field.
- This occurs wherever the **flow is parallel to geopotential height contours**.

- For such flows $\frac{DV}{Dt} = 0$ and F=ma for this flow reduces to the "gradient wind equation"

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

- Which is of course just the cross-flow ($\hat{n}$) component of the momentum equation.
- This is a **three-way force balance** between **centrifugal force** due to curvature of the flow, **Coriolis** force, and horizontal **PGF**.
- Define a Rossby number for balanced flow as $Ro = \left|\frac{V}{f\,R}\right|$
- **Ro is small** when the curvature is very large (flow is nearly in straight lines)
- For $Ro \ll 1$ the centrifugal force is negligible and we have **geostrophic balance** (again)

$$V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$$

- **Ro is large** for flows with small horizontal scale (large curvature), or fast characteristic time scales relative to Earth's rotation.
- For $Ro \gg 1$ the Coriolis force is negligible and we have **cyclostrophic balance** (e.g. in a tornado, or in a sink with a drain hole. The cyclostrophic wind is thus:

$$V = \sqrt{-R\frac{\partial \Phi}{\partial n}}$$

- Unlike geostrophic wind, the cyclostrophic wind around a low pressure has no preferred direction – it can be cyclonic or anticyclonic. Either way, the force balance is the same: PGF points radially inward, centrifugal force points radially outward.
- If **Ro is neither large nor small**, then we have to consider a **three-way force balance**. We call this the **gradient wind**. The gradient wind speed solution is

$$V = -\frac{fR}{2} \pm \sqrt{\left(\frac{fR}{2}\right)^2 - R\frac{\partial \Phi}{\partial n}}$$

- Using the definition of the geostrophic wind, we can also write this as

$$V = -\frac{fR}{2} \pm \sqrt{\left(\frac{fR}{2}\right)^2 + fRV_g}$$

- Several different solutions of the gradient wind are possible. We restrict ourselves to the solutions with **real, positive roots** (since V is defined as the wind speed, it can't be negative).
- **Balanced flow around a High** pressure/geopotential requires $\left|\frac{\partial \Phi}{\partial n}\right| < \frac{f^2|R|}{4}$
- Thus **the geopotential field has to flatten out** near the center of a High
- **There is no such constraint on a Low**. Pressure gradients and wind speeds can be large near the center of a Low.
- **See solutions to Homework 7** to review the classification of the different possible solutions of the gradient wind (**regular low**, **regular high**, **anomalous low**, **anomalous high**). Make sure you know how to **sketch the force balance** for each.

Thermal Wind

- Mid-latitude synoptic-scale motions are close to geostrophic, hydrostatic balance most of the time.
- Putting these two balances together (along with ideal gas law) gives us the thermal wind relation:
$$\frac{\partial \vec{v_g}}{\partial p} = -\frac{R}{fp}\hat{k}\times\nabla_p T$$
- Says that **horizontal temperature variations** must be accompanied by **vertical wind shear** (change in the horizontal wind with height/pressure). In component form:
$$\frac{\partial u_g}{\partial p} = \frac{R}{fp}\frac{\partial T}{\partial y}, \qquad \frac{\partial v_g}{\partial p} = -\frac{R}{fp}\frac{\partial T}{\partial x}$$
- E.g. if temperature decreases to the north, the wind must become more westerly with height in the NH.
- Horizontal temperature variations imply variations in thickness between pressure surfaces (hypsometric equation).
- Review and understand the sketch of vertical variations of the slope of pressure surfaces and implications for geostrophic wind. This is the essence of thermal wind relation.

Barotropic vs. Baroclinic fluids

- In **barotropic fluid**, density depends only on pressure, $\rho = \rho(p)$
- For ideal gas atmosphere, this means that **temperature is constant** on pressure surfaces.
- i.e. no horizontal temperature gradients.
- Thermal wind relation for barotropic fluid is just $\frac{\partial \vec{v_g}}{\partial p} = 0$
- Thus if there are no temperature gradients, the **geostrophic wind must be constant with height** – the whole column moves together.
- This phenomenon of "vertical rigidity" of a rotating fluid is called a **Taylor column**.
- A non-barotropic fluid is called **baroclinic**.
- Mid-latitude atmosphere is mostly baroclinic – temperature gradients are very common.
- Thus we find that winds vary with height almost everywhere.

The thermal wind vector and temperature advection

- Define thermal wind vector as vector difference between geostrophic wind vector at an upper level and a lower level, thus $\vec{V}_T = -\frac{\partial \vec{v_g}}{\partial p}$
- Thermal wind vector is parallel to lines of constant thickness (or column-averaged temperature), with lower thickness (colder air) to its left in the NH (right in the SH).
- Can infer the sign of geostrophic temperature advection just from knowing the direction of thermal wind.
    - Geostrophic winds veering with height (turning clockwise): warm air advection
    - Geostrophic winds backing with height (turning counter-clockwise): cold air advection
    - Can be determined from a single sounding.
- Review and understand sketch of column-average temperature advection.