ATM 500: Atmospheric Dynamics
Homework 3
Due Thursday October 1 2015

Corrected Wednesday September 23

1. *Question 2.1 in Vallis*

   Show that for an ideal gas in hydrostatic balance, changes in dry static energy ($M = c_pT + gz$) and potential temperature (θ) are related by $\delta M = c_p(T/\theta)\delta\theta$. (The quantity $c_p(T/\theta)$ is known as the 'Exner function', and is denoted Π.)

2. *Based on question 2.6 in Vallis*

   In a self-gravitating spherical fluid, like a star, hydrostatic balance may be written

   $$\frac{\partial p}{\partial r} = -\frac{GM(r)}{r^2}\rho$$

   where $M(r)$ is the mass interior to a sphere of radius r, and G is the universal gravitational constant. The radius of the star is a. Obtain an expression for the pressure as a function of radius, assuming that the fluid has a constant density.

3. Consider a cylindrical tank of water on a turntable, as we used on the first day of class. The depth of the water at any horizontal point is denoted $z = h(x, y)$, or $h(r, \theta)$ in polar coordinates (and $z = 0$ at the bottom of the tank). Assume that the water has constant density ρ_0 and is in hydrostatic balance.

   a. Derive an expression for the horizontal component of the pressure gradient force at any height z in the water column (with $z < h$) in terms of the gradient in the free surface height ∇h. Verify that your result shows that the pressure gradient force is

      - independent of height (i.e. the force is the same at every level in the column)
      - directed from higher to lower h.

      Note that the pressure at the top of the water column, $z = h$, is equal to the atmospheric pressure p_{atm}. You may assume that p_{atm} is the same at every point on the free surface.

   b. Now suppose the tank is rotating at a fixed angular rate Ω (radians s^{-1}). As we have seen, if we wait for a few minutes the tank will come into equilibrium in solid body rotation with a sloping surface. Assume a circular symmetry, so the only component of the height gradient ∇h is in the radial direction, dh/dr. Show that the slope of the surface at any horizontal distance r from the center of the tank must be

      $$\frac{dh}{dr} = \frac{\Omega^2 r}{g}$$

      in order for the horizontal pressure gradient force to balance the centrifugal force (in the rotating frame of reference).

c. Using your result from part *b.*, show that the free surface is parabolic in the radial direction.

d. The radius of the tank is r_0 and the depth of water was h_0 everywhere before we started the rotation. Now in solid body rotation at rate Ω, derive an expression for the height of the water surface as a function of radial distance $h(r)$ in terms of h_0. [Hint: consider the total volume of water in the tank]

e. If $r_0 = 1$ m and $h_0 = 30$ cm, calculate the rotation rate necessary to raise the water at the outer edge of the tank to h = $2h_0$. Express that rotation rate in two different ways:

- an angular rate, in radians s^{-1}
- a rotational period, i.e. the length of time required for a single rotation.

4. Consider an air parcel sitting at the equator, and at rest with respect to the rotating Earth.

   a. Calculate the absolute velocity of the air parcel as seen from outer space. The radius of the Earth is approximately 6.4×10^6 m.

   b. Calculate the centrifugal acceleration of the air parcel. Is it large or small relative to acceleration due to gravity?

   c. If the air parcel were located in the mid-latitudes rather than the equator, would its centrifugal acceleration be larger or smaller than what you just calculated? Why?

   d. What would the Earth's rotational period have to be (in hours) in order for the magnitude of the centrifugal acceleration at the equator to be equal to the magnitude of the gravitational acceleration?

5. *Question 2.7 in Vallis*

   At what latitude is the angle between the direction of Newtonian gravity (due solely to the mass of the Earth) and that of effective gravity (Newtonian gravity plus centrifugal terms) the largest? At what latitudes, if any, is this angle zero?