ATM 500: Atmospheric Dynamics
Homework 9
Due Thursday December 3 2015

Corrected Friday November 20

1. *Holton and Hakim, problem 4.1*

   What is the circulation about a square of 1000 km on a side for an easterly (i.e. westward-flowing) wind that decreases in magnitude toward the north at a rate of 10 m s^{-1} per 500 km? What is the mean relative vorticity in the square?

2. a. On an f-plane, show that the geostrophic velocity field on surfaces of constant pressure can be represented by a scalar streamfunction ψ with

   $$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

   and that the streamfunction is directly related to the geopotential Φ through

   $$\psi = \frac{\Phi}{f}$$

   b. Show that this relationship is not exact when f is allowed to vary.

   c. Show that relative vorticity ζ of the geostrophic wind field can be written

   $$\zeta = \nabla^2 \psi$$

   d. Suppose we have a geostrophic flow on some pressure surface in the northern hemisphere described by a streamfunction

   $$\psi(x, y) = -\psi_0 \exp\left(-\frac{x^2 + y^2}{L^2}\right)$$

   with ψ_0 some positive constant and L is a constant in length units.

   Draw a sketch of contours of ψ and the associated geostrophic wind vectors. Make sure your sketch clearly indicates the horizontal scales in terms of L.

   e. Calculate the relative vorticity $\zeta(x, y)$. Is is positive or negative near the center of the feature?

   f. Draw a sketch of the resulting ζ field.

   You might choose to plot this with a computer, which is fine. Just be sure to be clear about any assumptions you are making, e.g. the numerical value of L. You might also note that, since the field is isotropic (depending only on $x^2 + y^2$), you can treat this as a 1D field and just plot as a function of a single variable.

   g. Comment on the different spatial scales of your sketches, and what that might imply about the information content of a map of geopotential height versus a map of relative vorticity.

3. *You may want to read section 4.1 of the text before starting this question.*

Consider an axisymmetric vortex on a 2D plane. The natural way to describe such motion is with cylindrical coordinates (r, ϕ, z) where r is the radial distance from the center of the vortex, ϕ is the azimuthal angle, and z is the direction perpendicular to the plane. The velocity components can be written $\vec{v} = (u^r, u^\phi, u^z)$. Suppose that the flow is purely azimuthal and varies only with the radial distance, i.e. $u^z = u^r = 0$ and $u^\phi = u^\phi(r)$.

The vorticity vector is then strictly vertical:

$$\vec{\omega} = \nabla \times \vec{v} = \omega^z \hat{k}$$

where

$$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u^\phi\right)$$

in the component of the curl in cylindrical coordinates.

a. Show that the circulation C around a circle of radius R centered on the vortex is simply

$$C = 2\pi R U$$

where U is the azimuthal velocity at radius R, i.e. $u^\phi(R) = U$.

b. What does the circulation C tell you about the velocity field inside the circle? Give a formula for one possible velocity profile $u^\phi(r)$ that gives the correct circulation C. Draw a sketch of the flow (velocity) in this vortex between $r = 0$ and $r = R$.

c. Now repeat part b. for a *different* velocity profile $u^\phi(r)$ that has exactly the same circulation C. Make sure to draw another sketch to compare this to your first vortex.

d. Calculate the vorticity $\omega^z(r)$ for your two different cases.

e. Verify that the area integral of the vorticity is equal to C in both cases.

4. In class we wrote down an equation for the time-evolution of the vertical component of the vorticity vector:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

Scale analysis (e.g. Holton and Hakim, section 4.3.3) suggests that the divergence term is a more important source of vorticity than either the tilting term or the solenoidal term for mid-latitude synoptic-scale motion (though they can be important in smaller-scale motion such as fronts). So let's ignore those terms, and define $\delta = \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ to be the horizontal divergence. Then an approximate vorticity equation is

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\,\delta$$

a. Suppose that $\zeta = 0$ initially, and ignore changes in latitude. Which one will generate *cyclonic* relative vorticity: horizontal *convergence* or horizontal *divergence*? Conversely, which one will generate *anticyclonic* relative vorticity? Does your answer depend on which hemisphere we are in? Draw sketches to illustrate your argument.

b. Suppose that δ is constant in time. Does the rate of change of vorticity get larger or smaller with time? What happens when the magnitude of the relative vorticity approaches the magnitude of the planetary vorticity? Answer separately for the cyclone and anticyclone. (Again, suppose that there are no changes in latitude).

c. Discuss the potential significance of your findings for the maximum intensity of cyclones versus anticyclones.

5. How should the absolute vorticity of an air parcel change if its static stability increases? Why?