ATM 500: Atmospheric Dynamics

End-of-semester review, December 8 2015

Here, in bullet points, are the key concepts from the second half of the course. They are not meant to be self-contained notes! Refer back to your course notes for notation, derivations, and deeper discussion.

Static stability The vertical variation of density in the environment determines whether a parcel, disturbed upward from an initial hydrostatic rest state, will accelerate away from its initial position, or oscillate about that position.

For an incompressible or Boussinesq fluid, the buoyancy frequency is

$$N^2 = -\frac{g}{\tilde{\rho}}\frac{d\tilde{\rho}}{dz}$$

For a compressible ideal gas, the buoyancy frequency is

$$N^2 = \frac{g}{\tilde{\theta}}\frac{d\tilde{\theta}}{dz}$$

(the expressions are different because density is not conserved in a small upward displacement of a compressible fluid, but potential temperature is conserved)

Fluid is *statically stable* if $N^2 > 0$; otherwise it is *statically unstable*.

Typical value for troposphere: $N = 0.01$ s^{-1}, with corresponding oscillation period of 10 minutes.

Dry adiabatic lapse rate

$$\Gamma_d = \frac{g}{c_p} = 9.8 \text{ K km}^{-1}$$

The rate at which temperature decreases during adiabatic ascent (for which $\frac{D\theta}{Dt} = 0$)

"Dry" here means that there is no latent heating from condensation of water vapor.

Static stability criteria for a dry atmosphere Environment is stable to dry convection if

$$\frac{d\tilde{\theta}}{dz} > 0 \quad \text{or} \quad -\frac{d\tilde{T}}{dz} < \Gamma_d$$

and otherwise unstable.

Buoyancy equation for Boussinesq fluid with background stratification

$$\frac{Db'}{Dt} + N^2 w = 0$$

where we have separated the full buoyancy into a mean vertical part $\hat{b}(z)$ and everything else (b'), and $N^2 = \frac{d\hat{b}}{dz}$. We often assume that N^2 is a fixed background stratification and study have the motion depends on N^2.

Recipe for wave analysis Here is a generic recipe for analyzing wavelike motions in a fluid. The key idea is that we *linearize* the equations for small perturbations away from a known reference state.

1. Choose a set of governing equations.
2. Choose a reference state that satisfies those equations.
3. Expand all variables in perturbations away from the reference state:

$$\vec{v} = \vec{v}_0 + \vec{v}'$$

etc.

4. Substitute expanded variables into equation, subtract out the reference state. *Neglect all products of small primed quantities.*
5. Assume a wavy solution with unknown frequency and wavenumbers.
6. Plug into equation and derive the conditions for which the wavy solution is valid – a relationship between frequency, wavenumbers, and physical parameters of the fluid known as the *dispersion relation.*

Phase speed The speed at which individual peaks and troughs travel in the three coordinate directions is

$$c^x = \frac{\omega}{k} \quad c^y = \frac{\omega}{l} \quad c^z = \frac{\omega}{m}$$

Dispersive vs. non-dispersive waves A wave is *non-dispersive* if the phase speed *does not depend on wavenumber.* In this case long and short waves all travel at the same speed, so a wave packet (the superposition of many different wavelengths) remains coherent as it travels. All other waves are *dispersive.*

Group velocity A vector with three components

$$\vec{c_g} = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m} \right)$$

Gives the speed and direction of the propagation of the wave *envelope*, which is also the speed at which energy is carried by the wave.

For a non-dispersive wave the components of $\vec{c_g}$ are just the phase speeds.

Internal gravity waves Linearize the non-rotating Boussinesq equations with background stratification N^2 about a motionless hydrostatic reference state. Get the dispersion relation

$$\omega = \pm \frac{kN}{\sqrt{k^2 + m^2}}$$

Dispersive waves that propagate equally in two directions, with long waves propagating faster than short waves.

Statically unstable environment If $N^2 < 0$ then the above analysis shows that a small disturbance *grows exponentially* with growth rate

$$\pm \frac{k\sqrt{-N^2}}{\sqrt{k^2 + m^2}}$$

which gives a simple model for convection in an unstable environment.

Force balance in frictional boundary layer Three-way force balance between PGF, Coriolis force and friction (assumed to act in the direction of $-\vec{u}$) requires flow across isobars toward the low pressure.

Ekman layer A turbulent, frictional boundary layer in which we assume

- Boussinesq approximation (density variations are small)
- Finite depth of frictional effects, smaller than total depth of fluid
- Steady, hydrostatic motion

We integrate the horizontal momentum equation over the Ekman layer to get the mass transport

$$\vec{M_E} = \frac{1}{f} \hat{k} \times \vec{\tau}_S$$

for the atmospheric Ekman layer, or

$$\vec{M}_E = -\frac{1}{f}\hat{k} \times \vec{\tau}_S$$

for the ocean top Ekman layer, where in both cases $\vec{\tau}_S$ is the stress exerted by the atmosphere on the surface.

These transports are equal and opposite, which is a consequence of Newton's laws of motion.

Ekman pumping Convergence or divergence of the Ekman mass transport must be accompanied by vertical motion (to conserve mass). This is called *Ekman pumping*, given by

$$w_E = \frac{1}{\rho_0} curl_z \left(\frac{\vec{\tau}_s}{f}\right)$$

where w_E is the frictionally-induced vertical velocity, either at the top of the atmospheric Ekman layer, or the bottom of the oceanic Ekman layer.

For the atmosphere, we infer that air is pumped upward by surface friction near the center of a low pressure system and sucked downward near the center of a high pressure system.

In the ocean, the stress is mainly determined by the overlying wind. Ekman pumping pushes fluid in or out of the oceanic interior, setting the inner ocean in motion far from the direct influence of the surface friction.

Shallow water equations Assume that a shallow layer of fluid is in hydrostatic balance with a constant density ρ_0, then integrate vertically. Then horizontal pressure gradients are the same at every height, and are determined by variations in the free surface height.

The equations of motion then simplify to a two-dimensional system:

$$\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -g\nabla\eta$$
$$\frac{Dh}{Dt} + h\nabla \cdot \vec{u} = 0$$

where η is the height of the free surface and h is the full depth at every point. (For a flat-bottomed fluid $\eta = h$).

Relative and absolute vorticity In shallow water we can define

$$\zeta = curl_z \vec{u} = \hat{k} \cdot \nabla \times \vec{u} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

which is the *relative vorticity* – a measure of the local spin of the fluid. The *absolute vorticity* is $\zeta + f$ including the planetary vorticity f.

Taking the curl of the momentum equation gives an equation for the absolute vorticity in shallow water:

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\nabla \cdot \vec{u}$$

Conservation of shallow-water potential vorticity Combining the vorticity equation with continuity gives a conservation equation

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0$$

The quantity $Q = \frac{\zeta+f}{h}$ is called the shallow-water potential vorticity. It is conserved for parcels in the shallow water system.

Cyclonic versus anticyclonic vorticity Motion for which ζ has the same sign as f is called "cyclonic". The relative spin is in the same direction as the Earth's rotation. For $f > 0$ (northern hemisphere) cyclonic means anti-clockwise. The opposite spin is called anticyclonic.

Vortex stretching Conservation of PV in shallow water implies that any stretching or squashing of a fluid column (changes in height) must be associated with a change in the total rotation of the column – either change in relative vorticity, change in latitude, or both.

Stretching out a column tends to increase its vorticity, in analogy with a figure skater pulling his/her arms in to rotate faster.

Non-rotating shallow-water waves Linearize the non-rotating shallow-water equations for small perturbations about a state of rest with flat bottom and a mean depth H, get a dispersion relation

$$\omega = \pm\sqrt{gH}k$$

Non-dispersive surface gravity waves with a phase speed $\sqrt{gH}$ depending on the depth. Waves do not have a preferred direction of propagation.

Rotating shallow-water waves Repeat the analysis on a f-plane. System now has the possibility of a steady solution with non-zero variations in the free surface height, because Coriolis force can balance PGF. The dispersion relation has three roots:

$$\omega = 0$$

corresponds to the steady, geostrophically balanced solution. Otherwise

$$\omega^2 = f_0^2 + gHK^2$$

with $K^2 = (k^2 + l^2)$ the horizontal wavenumber. Known as Poincaré waves, surface gravity waves modified by rotation. Unlike non-rotating case, these waves are *dispersive*.

The short-wave limit is $\omega^2 = gHK^2$: short waves behave just like non-rotating waves.

The long-wave limit is $\omega = \pm f_0$: inertial oscillations.

Deformation radius

$$L_d = \frac{\sqrt{gh}}{f}$$

is a length scale that separates "short" from "long" in the shallow water system. Waves with scales similar to L_d have properties of both rotating and non-rotating motion.

Adjustment problem – non-rotating Start with initially discontinuous free surface with "top hat" shape. Non-rotating fluid is unbalanced: there are pressure forces with nothing to balance them. There must be accelerations. Because non-rotating waves are non-dispersive, surface fronts propagate coherently away toward infinity. Initial disturbance radiates completely away, leaving no memory of itself at the initial location.

Geostrophic adjustment With rotation, final adjusted state has non-zero height variations because these can be balanced by Coriolis force. Gravity waves radiate away from initial location, but the initial condition is not completely forgotten. Use conservation of PV to find the final adjusted state: same PV as the initial state, but velocity and height fields in geostrophic balance. Find the free surface adjusts *smoothly* between the initial heights over a characteristic distance L_d.

The deformation radius represents the *smallest scales that can be geostrophically balanced*. Any variations in the free surface over scales smaller than L_d will get smoothed out in the adjustment process.

Vorticity is defined as the curl of the velocity field

$$om\vec{e}ga = \nabla \times \vec{v}$$

It is a measure of the local spin of the fluid. The vector ω at any point is the rotation vector that an infinitesimal paddle wheel would acquire if it were placed in the fluid at that point. The direction of ω indicates the axis of rotation through the right hand rule.

Circulation is defined as the line integral of the velocity field around any closed fluid loop:

$$C = \oint \vec{v} \cdot d\vec{r}$$

Using Stokes' theorem, the circulation is also the area-integral of vorticity:

$$C = \int_S \vec{\omega} \cdot d\vec{S}$$

where S is a surface bounded by the closed loop.

Circulation is a scalar, and its value depends on the particular path chosen (it is not a field variable like velocity, vorticity, temperature, etc.)

Vorticity equation Take the curl of the 3D momentum equation in the inertial frame to get

$$\frac{D\vec{\omega}}{Dt} = \vec{\omega} \cdot \nabla\vec{v} - \vec{\omega}\nabla \cdot \vec{v} + \frac{1}{\rho^2}\left(\nabla\rho \times \nabla p\right) + \nabla \times \vec{F}$$

where F represents any body forces including gravity and friction.

Alternatively we can combine this with the continuity equation to get

$$\frac{D\vec{\tilde{\omega}}}{Dt} = \vec{\tilde{\omega}} \cdot \nabla\vec{v} + \frac{1}{\rho^3}\left(\nabla\rho \times \nabla p\right) + \frac{1}{\rho}\nabla \times \vec{F}$$

with $\vec{\tilde{\omega}} = \vec{\omega}/\rho$.

The term $\nabla\rho \times \nabla p$ is called the solenoidal term, and is zero for a barotropic fluid.

Frozen-in property of vorticity A vortex line is any line drawn through the fluid which is everywhere in the direction of the local vorticity vector. For an unforced, inviscid, *barotropic* fluid, a vortex line is always coincident with the same material line. The vorticity is frozen or glued to material elements. Any stretching or deformation of material lines thus must result in changes in the vorticity.

Sources and sinks of vorticity due to deformation of material lines by the flow can be divided into tilting and stretching terms.

Tilting or tipping terms If advection acts to tilt material lines, the vorticity vector changes orientation. Vertical component of $\vec{\omega}$ can project onto horizontal and vice-versa.

Stretching term Vorticity increases in direction that material lines are stretched out. Same as vortex stretching in shallow water, but generalized to three dimensions.

Kelvin's circulation theorem For a *barotropic* fluid, the circulation is conserved:

$$\frac{DC}{Dt} = 0$$

This is related to the narrowing of a vortex tube as it is stretched. The vorticity increases from the stretching, but material lines are drawn closer together so the area integral of the vorticity is conserved.

Absolute and relative vorticity in the rotating frame Absolute vorticity is

$$\vec{\omega}_a = \vec{\omega}_r + 2\vec{\Omega}$$

where $\vec{\omega}_r = \nabla \times \vec{v}_r$ is the relative vorticity and $2\vec{\Omega}$ is the vorticity due to the solid-body rotation of the planet.

Circulation theorem in rotating frame For a rotating, barotropic fluid

$$\frac{D}{Dt} \int_S \left(\vec{\omega}_r + 2\vec{\Omega} \right) \cdot d\vec{S} = 0$$

Vorticity equation in rotating frame

$$\frac{D\vec{\omega}_r}{Dt} = \left(2\vec{\Omega} + \vec{\omega}_r \right) \cdot \nabla \vec{v} - \left(2\vec{\Omega} + \vec{\omega}_r \right) \nabla \cdot \vec{v} + \frac{1}{\rho^2} \left(\nabla \rho \times \nabla p \right)$$

(neglecting the frictional term)

Vertical component of absolute vorticity Vertical component is

$$\hat{k} \cdot \left(2\vec{\Omega} + \vec{\omega}_r\right) = f + \zeta, \quad \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

The vertical component of relative vorticity evolves according to

$$\frac{D\zeta}{Dt} = -\beta v - (f + \zeta)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial u}{\partial z}\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\frac{\partial w}{\partial x}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

There are thus 4 different possible sources or sinks of relative vorticity for a parcel, numbering the terms on the RHS from 1 to 4:

1. Beta effect: if a parcel moves south, its relative vorticity must increase.
2. Divergence term, or vortex stretching term. If w increases with height, flow will be horizontally convergent, column is stretched out, and relative vorticity increases.
3. Tilting term: vertical vorticity can be generated by vertical velocity acting on horizontal vorticity.
4. Solenoidal term: arises when surface of pressure and density are not parallel (baroclinic fluid).

Conservation of vorticity on tracer surfaces For any conserved tracer χ, if the frozen-in property of vorticity applies on surfaces of constant χ, there is a conservation law

$$\frac{D}{Dt}\left(\vec{\tilde{\omega}}_a \cdot \nabla\chi\right) = 0$$

This is a conservation law for the component of the absolute vorticity normal to surfaces of constant χ – i.e. a constraint on the *velocity* components along the surface of constant χ.

This result only holds for tracers for which there is *no baroclinicity* on tracer surfaces, i.e. $\rho = \rho(p)$ on those surfaces.

Ertel's potential vorticity The potential temperature θ is a natural choice for the tracer since

- it is conserved for adiabatic motion
- There is no baroclinicity on surfaces of constant θ (the solenoidal term $\nabla\rho \times \nabla p$ vanishes on these surfaces)

Ertel's theorem thus says

$$\frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla\theta}{\rho}\right) = 0$$

This is an exact result for any adiabatic, inviscid flow, and links together all the basic physical laws (momentum, mass, thermodynamics) into a single scalar conservation equation.

The quantity $\frac{\vec{\omega}\cdot\nabla\theta}{\rho}$ is called Ertel's potential vorticity. It is measured in "PV units" or "PVU", with 1 PVU defined as 10^{-6} K m^2 s^{-1} kg^{-1}.

PV equation with non-conservative terms The fully general PV equation, valid in the presence of diabatic heating and friction, looks like

$$\frac{D}{Dt}\left(\frac{\vec{\omega}\cdot\nabla\theta}{\rho}\right) = \frac{\vec{\omega}\cdot\nabla\dot{\theta}}{\rho} + \frac{\nabla\theta}{\rho}\cdot\left(\nabla\times\vec{F}\right)$$

where the potential temperature equation is

$$\frac{D\theta}{Dt} = \dot{\theta}$$

Non-dimensional shallow-water equations Take the shallow-water equations are write them in terms of non-dimensional variables:

$$Ro\left(\frac{\partial\vec{\hat{u}}}{\partial\hat{t}} + \vec{\hat{u}}\cdot\nabla\vec{\hat{u}}\right) + \vec{\hat{f}}\times\vec{\hat{u}} = -\nabla\hat{\eta}$$

$$Ro\left(\frac{L}{L_d}\right)^2\frac{D\hat{\eta}}{D\hat{t}} + \left(1 + Ro\left(\frac{L}{L_d}\right)^2\hat{\eta}\right)\nabla\cdot\vec{\hat{u}} = 0$$

with $Ro = \frac{U}{f_0 L}$ the Rossby number and $L_d = \frac{\sqrt{gH}}{f_0}$ the deformation radius in shallow water.

The non-dimensional form allows a careful scaling of the equations under various dynamical regimes, so we can be consistent about which terms can be neglected.

Here $Ro\left(\frac{L}{L_d}\right)^2$ is the scaling for variations in the free surface height, relative to the mean fluid depth H. If this number is small, the continuity equation becomes simply $\nabla\cdot\vec{\hat{u}} = 0$ (horizontal non-divergence).

Deformation radius for stratified fluid The equivalent scaling in a stratified fluid is the ratio of vertical variations in buoyancy relative to background stratification. In the Boussinesq system with background stratification N^2 this ratio is

$$Ro\frac{L^2}{\left(\frac{NH}{f_0}\right)^2}$$

which reveals the equivalent of the deformation radius for the stratified fluid

$$L_d = \frac{NH}{f_0}$$

For the mid-latitude atmosphere, we find $N \approx 0.01$ s^{-1} (period of 10 minutes), and $L_d \approx 1000$ km. Thus for synoptic-scale systems, L/L_d is order-1.

Quasi-geostrophic approximation Derive a simplified set of equations to describe evolution of the *geostrophic* flow while eliminating faster, smaller-scale motions (gravity waves).

Starting from the shallow water equations, make the following assumptions:

1. $Ro << 1$ (flow near geostrophic balance)
2. $Ro\frac{L^2}{\left(\frac{NH}{f_0}\right)^2}$ is of the same order as Ro (implying variations in free surface are small (shallow water) or variations in stratification are weak (stratified fluid))
3. Variations in f are small (limiting to modest north-south variations)

Formally expand the variables in power series using the Rossby number Ro as the small parameter: $u = u_0 + Ro\ u_1 + Ro^2\ u_2 + ...$, etc. Require that the equations hold independently at every order of Ro.

Leading order in Rossby number expansion is simply geostrophic balance on an f-plane:

$$f_o\hat{k} \times \vec{\hat{u_0}} = -\nabla\hat{\eta_0}$$

where this zero-order wind field is exactly non-divergent:

$$\nabla \cdot \vec{\hat{u_0}} = 0$$

Importantly, this is simply a balance statement and does give a prognostic equation (needed to make predictions).

Geostrophic vorticity equation Taking the curl of the order Ro terms in the momentum equation gives

$$\frac{\partial \hat{\zeta}_0}{\partial \hat{t}} + \vec{\hat{u}_0} \cdot \nabla \left(\hat{\zeta}_0 + \hat{\beta} \hat{y} \right) = -f_0 \nabla \cdot \vec{\hat{u}_1}$$

which says that the geostrophic vorticity evolves due to the beta effect, and also due to vortex stretching associated with the small ageostrophic divergence / convergence.

Quasi-geostrophic potential vorticity The divergence is related to free-surface changes through mass continuity. Together these give an equation that is closed in terms of 0-order geostrophic quantities. It is a conservation equation for the non-dimensional quantity

$$\nabla^2 \hat{\psi}_0 + \hat{\beta} \hat{y} - f_0^2 \left(\frac{L}{L_d} \right)^2 \hat{\psi}_0$$

where ψ_0 is a streamfunction for the geostrophic velocity:

$$\hat{u}_0 = -\frac{\partial \hat{\psi}_0}{\partial \hat{y}}, \quad \hat{v}_0 = \frac{\partial \hat{\psi}_0}{\partial \hat{x}}$$

where the advecting velocity is the same order-0 geostrophic velocity.

Conservation of shallow water QGPV – dimensional form Under the quasi-geostrophic conditions assumed above, the flow obeys

$$\frac{D_g q}{Dt} = 0$$

where

$$q = \zeta + \beta y - \frac{f_0}{H} \eta = \nabla^2 \psi + \beta y - \frac{\psi}{L_d^2}$$

is the shallow-water quasi-geostrophic potential vorticity.

The subscript $_g$ in the material derivative is a reminder that the advection is strictly by the geostrophic velocity.

Inverting the QGPV Since q is advected by the geostrophic velocity, and the geostrophic velocity in turn can be computed from q, the QGPV contains all the information about its own evolution. We can thus construct a predictive model for the flow by iterating through these steps:

1. Take the height field η at time 0, calculate geostrophic wind, vorticity, streamfunction, and potential vorticity.
2. Step q forward in time using the QGPV equation:

$$\frac{\partial q}{\partial t} = \frac{\partial \psi}{\partial y}\frac{\partial q}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial q}{\partial y}$$

   where the terms on the RHS are all known at time 0. Get a prediction for q at future time 1.
3. Solve the boundary value problem

$$\nabla^2 \psi - \frac{\psi}{L_d^2} + \beta y = q$$

   for $\psi(x, y)$ at time 1.
4. Use the new value of ψ to compute the advective velocities at time 1.
5. Go back to step 2.

Step 3, in which we solve for the velocity and height field using the geostrophic relationships, is known as "inverting the PV".

The beta effect Displace a motionless material line north and south of its resting latitude. Relative vorticity increases where fluid is displaced to the south and vice-versa. There is now a non-zero velocity acting to advect the material line. Net effect is advection of the peaks and troughs *to the west*.

So the disturbance will propagate to the west because the planetary vorticity increases to the north. The restoring force is the background PV gradient, which we measure through the parameter β.

Rossby waves in shallow water Consider layer of shallow water on flat-bottomed beta plane obeying the QG dynamics. Consider the simplest case $L << L_d$ (valid for short waves). Linearize the QGPV equation about a basic state with a constant background zonal wind U. We find a single wavy solution with dispersion relation

$$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

These are dispersive waves. The phase speed in the x direction is

$$c^x = \frac{\omega}{k} = U - \frac{\beta}{k^2 + l^2}$$

which is always to the west relative to the mean flow. Longer waves travel westward faster than shorter waves.

Taking a typical wavelength for synoptic scale waves as $k = l = 6000$ km, we get waves that travel westward at 7.5 m s^{-1} relative to the mean flow. Since the mean westerly wind U is usually greater than this, most waves will be observed to travel eastward, but at a slower speed than U. Very long waves might be stationary with respect to the ground.

The group velocity in the x-direction is

$$c_g^x = \frac{\partial \omega}{\partial x} = U + \beta \frac{k^2 - l^2}{k^2 + l^2}$$

which can be either positive or negative. For the above example with $k = l$ we have simply $c_g^x = U$. The group velocity is thus eastward, and faster than the phase speed. New disturbances can thus be expected to develop downstream of existing disturbances.