ATM 500: Atmospheric Dynamics
Homework 10
Due Tuesday November 29 2016

1. Read section 3.7.3 in Vallis. The *Kelvin wave* is a special kind of gravity wave that exist in the presence of a lateral boundary. The wave is in geostrophic balance in the direction perpendicular to the boundary, but unbalanced (and propagating at phase speed $\sqrt{gH}$) in the direction along the boundary. (These waves are particularly important for our understanding of tropical phenomena such as El Niño).

   Why did we not find the Kelvin wave when we did our standard wave analysis of the rotating shallow water equations in class?

2. Starting from the conservation of potential vorticity $Q = (f + \zeta)/h$ for parcels in the shallow water system, derive the linear result

$$\frac{\partial q}{dt} = 0, \qquad q = \zeta' - f_0 \frac{\eta'}{H}$$

   for small perturbations of the height and velocity field on an f-plane with a flat bottom.

3. *In this question you will investigate a variant of the geostrophic adjustment problem using the shallow water equations. Here the initial imbalance exists because of a discontinuity in the velocity field rather than in the free surface.*

   a. Suppose we have a infinite slab of shallow water on an f-plane. Initially, the fluid surface is flat, the zonal velocity u is zero everywhere, but there is a meridional velocity given by

$$v(x) = v_0 \,\mathrm{sgn}(x)$$

   where v_0 is a constant. This means that the fluid is moving northward at a speed v_0 everywhere to the east of $x = 0$, and southward at speed v_0 everywhere to the west of $x = 0$.

   Verify that this initial condition is NOT a steady solution of the shallow water equations (i.e. show that there must be a non-zero time tendency of velocities or surface height or both, at least somewhere in the domain).

   b. Show that $q = 0$ everywhere in the domain *except* at $x = 0$.

   c. After the adjustment the system will reach a final steady state that can be described with a geostrophic streamfunction $\psi(x, y)$ satisfying

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

   Since the linearized PV field is fixed in space, the final steady state is the solution of this equation:

$$\frac{d^2\psi}{dx^2} - \frac{\psi}{L_d^2} = q$$

where q is the initial distribution of PV that you worked out above.

Solve the differential equation for ψ separately for $x > 0$ and $x < 0$. Apply a physically sensible boundary condition for $x \to \pm\infty$. Be sure to explain your reasoning. There should be one unknown constant left in each solution.

d. Because of the discontinuity in the initial velocity field at $x = 0$, the initial relative vorticity ζ is locally infinite at $x = 0$, and therefore $q(x)$ is also. We can describe this with a so-called 'delta function'

$$q(x) = 2v_0\delta(x)$$

where $\delta(x) = 0$ by definition everywhere except $x = 0$, but it also has the property that if we integrate over any interval, the result is finite and independent of the interval:

$$\int_{x_1}^{x_2} \delta(x)dx = 1$$

so long as the interval contains $x = 0$ (i.e. $x_1 < 0$ and $x_2 > 0$).

To evaluate your unknown constants and complete your solution, apply these two boundary conditions:

- η' is continuous at the boundary in the final state.
- The integral $\int_{-\infty}^{+\infty} q\ dx$ must be the same in the initial and final states.

Using these conditions, what is $\psi(x)$?

e. Solve for the final surface height and velocity fields. Draw a sketch of the final adjusted state. Is there anything unusual or perhaps unphysical about the final state?