ATM 500: Atmospheric Dynamics
Homework 9
Due Tuesday November 28 2017

1. *In this question you will investigate a variant of the geostrophic adjustment problem using the shallow water equations. Here the initial imbalance exists because of a discontinuity in the velocity field rather than in the free surface or pressure field.*

   a. Suppose we have a infinite slab of shallow water on an f-plane. Initially, the fluid surface is flat, the zonal velocity u is zero everywhere, but there is a meridional velocity given by
   $$v(x) = v_0 \,\mathrm{sgn}(x)$$
   where v_0 is a constant. This means that the fluid is moving northward at a speed v_0 everywhere to the east of $x = 0$, and southward at speed v_0 everywhere to the west of $x = 0$.

      Verify that this initial condition is NOT a steady solution of the shallow water equations (i.e. show that there must be a non-zero time tendency of velocities or surface height or both, at least somewhere in the domain).

   b. Show that $q = 0$ everywhere in the domain *except* at $x = 0$.

   c. After the adjustment the system will reach a final steady state that can be described with a geostrophic streamfunction $\psi(x, y)$ satisfying
   $$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$
   Since the linearized PV field is fixed in space, the final steady state is the solution of this equation:
   $$\frac{d^2\psi}{dx^2} - \frac{\psi}{L_d^2} = q$$
   where q is the initial distribution of PV that you worked out above.

      Solve the differential equation for ψ separately for $x > 0$ and $x < 0$. Apply a physically sensible boundary condition for $x \to \pm\infty$. Be sure to explain your reasoning. There should be one unknown constant left in each solution.

   d. Because of the discontinuity in the initial velocity field at $x = 0$, the initial relative vorticity ζ is locally infinite at $x = 0$, and therefore $q(x)$ is also. We can describe this with a so-called 'delta function'
   $$q(x) = 2v_0\delta(x)$$
   where $\delta(x) = 0$ by definition everywhere except $x = 0$, but it also has the property that if we integrate over any interval, the result is finite and independent of the interval:
   $$\int_{x_1}^{x_2} \delta(x)dx = 1$$

so long as the interval contains $x = 0$ (i.e. $x_1 < 0$ and $x_2 > 0$).

To evaluate your unknown constants and complete your solution, apply these two boundary conditions:

- η' is continuous at the boundary in the final state.
- The integral $\int_{-\infty}^{+\infty} q \, dx$ must be the same in the initial and final states.

Using these conditions, what is $\psi(x)$?

e. Solve for the final surface height and velocity fields. Draw a sketch of the final adjusted state. Is there anything unusual or perhaps unphysical about the final state?

2. *Holton and Hakim, problem 4.1*

   What is the circulation about a square of 1000 km on a side for an easterly (i.e. westward-flowing) wind that decreases in magnitude toward the north at a rate of 10 m s^{-1} per 500 km? What is the mean relative vorticity in the square?

3. An approximate vorticity equation for mid-latitude synoptic scale motion is

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\,\delta$$

where $\delta = \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the horizontal divergence. (This follows from scale analysis of the full vorticity equation, e.g. Holton and Hakim, section 4.3.3, which shows that the divergence term is a more important source of vorticity than either the tilting term or the solenoidal term for such scales).

a. Suppose that $\zeta = 0$ initially, and ignore changes in latitude. Which one will generate *cyclonic* relative vorticity: horizontal *convergence* or horizontal *divergence*? Conversely, which one will generate *anticyclonic* relative vorticity? Does your answer depend on which hemisphere we are in? Draw sketches to illustrate your argument.

b. Suppose that δ is constant in time. Does the rate of change of vorticity get larger or smaller with time? What happens when the magnitude of the relative vorticity approaches the magnitude of the planetary vorticity? Answer separately for the cyclone and anticyclone. (Again, suppose that there are no changes in latitude).

c. Discuss the potential significance of your findings for the maximum intensity of cyclones versus anticyclones.

4. How should the absolute vorticity of an air parcel change if its static stability increases? Why?