ATM 500: Atmospheric Dynamics

Midterm review, October 16 2017

Here, in bullet points, are the key concepts from the first half of the course. They are not meant to be self-contained notes! Refer back to your course notes for notation, derivations, and deeper discussion.

Advection is the transport of properties by the movement of fluid parcels. The local rate of change of any quantity A due to advection is

$$-\vec{v} \cdot \nabla A$$

Material Derivative expresses the rate of change of any quantity in the Lagrangian framework, moving with the fluid.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \nabla$$

i.e. the difference between the Eulerian change (at fixed points) and the advection.

Most of the laws of physics are expressed as conservation statements for parcels of fluid, so are most naturally written in the Lagrangian form with material derivatives.

Conservation of mass Conservation of mass for individual fluid parcels is expressed, for a moving continuum, in either Eulerian form as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

or in Lagrangian form

$$\frac{1}{\rho} \frac{D\rho}{Dt} + \nabla \cdot \vec{v} = 0$$

Fundamental forces "Fundamental" forces are those that act on parcels in all reference frames, including the inertial frame. We almost always express forces on fluids *per unit mass*, so the forces have units of acceleration.

Pressure gradient force $-\frac{1}{\rho} \nabla p$

Gravitational force $-g\hat{k}$

Viscous force $\nu \nabla^2 \vec{v}$

Momentum equation Newton's second law for a moving fluid: acceleration of parcels = net force per unit mass

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -g\hat{k} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}$$

Hydrostatic balance A balance between the vertical component of the pressure gradient force and gravity (or more precisely, effective gravity for a rotating fluid on a sphere, see below).

$$\frac{\partial p}{\partial z} = -\rho g$$

Equation of state Every fluid has an equation of state, which is a diagnostic (not prognostic) relationship between the thermodynamic state variables for that fluid. For the atmosphere we usually use the ideal gas law

$$p = \rho R_d T$$

with R_d the gas constant for dry air.

Barotropic vs. baroclinic fluid A fluid is barotropic if $\rho = \rho(p)$, i.e. there is a one-to-one relationship between density and pressure. Otherwise the fluid is baroclinic and the density depends on another state variable, e.g. $\rho = \rho(p, T)$.

An ideal gas is barotropic if temperature is constant on surfaces of constant pressure.

Scale height Assume temperature is vertically uniform, then integrate the hydrostatic relation vertically to get

$$\rho = \rho_0 \exp\left(-\frac{z}{H}\right), \qquad p = p_0 \exp\left(-\frac{z}{H}\right)$$

where $H = \frac{R_d T}{g} \approx 8$ km is the scale height for the atmosphere (approximate e-folding scale for vertical variations of density and pressure).

1st law of thermodynamics The sum of internal energy changes and work done must equal the external heating rate $\dot{Q}$.

$$\frac{DI}{Dt} + p\frac{D\alpha}{Dt} = \dot{Q}$$

For an ideal gas, internal energy is proportional to temperature, $dI = c_v dT$. So the first law can also be written

$$c_v \frac{DT}{dt} + p\alpha \nabla \cdot \vec{v} = \dot{Q}$$

Potential temperature Define potential temperature θ as

$$\theta = T \left(\frac{p_{ref}}{p} \right)^{\kappa}$$

Then changes in temperature due to pressure-work upon expansion or contraction are accounted for, and we can write the 1st law as

$$c_p \frac{D\theta}{Dt} = \frac{\theta}{T} \dot{Q}$$

Adiabatic processes If there is no external energy source ($\dot{Q} = 0$) then potential temperature θ is conserved:

$$\frac{D\theta}{Dt} = 0$$

This would be true e.g. for a freely ascending parcel of air. Temperature decreases due to pressure decrease (adiabatic expansion) while potential temperature remains constant.

Sound waves Fast wave motion associated with adiabatic expansions and contractions. The restoring force is the compressibility of the fluid. Dispersion relation is $\omega = \pm c_s k$, where the phase speed is given by

$$c_s^2 = \frac{c_p}{c_v} RT$$

Energy budget Kinetic energy is $K = \vec{v} \cdot \vec{v}$, so to get evolution equation for K, take $\vec{v}\cdot$ (momentum equation).

Total energy equation for a compressible adiabatic fluid is

$$\frac{\partial}{\partial t} E + \nabla \cdot [\vec{v}\,(E + p)] = 0$$

where $E = \rho \left(\frac{1}{2} v^2 + I + \Phi \right)$ is the total energy per unit volume of the fluid, I is the internal energy, and $\Phi \approx gz$ is the potential for gravity plus any other conservative force per unit mass.

The energy flux contains the term $\vec{v}p$ which is the work done against pressure. Total energy is thus conserved only over a closed domain with rigid boundaries.

Inertial vs. non-inertial frames A non-inertial frame of reference is one in which the coordinates are *accelerating*. The acceleration of the reference frame introduces apparent forces. The rotating system is one example of a non-inertial frame.

Rate of change of vectors in rotating frame Here $\vec{\Omega}$ is the rotation vector for the reference frame. The general transformation (subscripts indicate "inertial" and "rotating"):

$$\left(\frac{d\vec{B}}{dt}\right)_I = \left(\frac{d\vec{B}}{dt}\right)_R + \vec{\Omega} \times \vec{B}$$

Applying this transformation, the relationship between velocities in the two frames are

$$\vec{v}_I = \vec{v}_R + \vec{\Omega} \times \vec{r}$$

and for the accelerations (need to apply the transformation twice):

$$\left(\frac{d\vec{v}_R}{dt}\right)_R = \left(\frac{d\vec{v}_I}{dt}\right)_I - 2\vec{\Omega} \times \vec{v}_R - \vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right)$$

Apparent forces Because of the coordinate transformation, two non-inertial forces appear in the rotating momentum equation:

Centrifugal force

$$\vec{F}_{Ce} = -\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = \Omega^2 \vec{r}_\perp$$

acts radially outward from axis of rotation

Coriolis force

$$\vec{F}_{Co} = -2\vec{\Omega} \times \vec{v}_R$$

acts at right angles to the *relative* velocity field $\vec{v}_R$. (Think of it as "excess centrifugal force" for the parcel moving at a different speed than the Earth).

$\vec{F}_{Co}$ does no work since it is always directed perpendicular to the motion.

Effective gravity We define the geopotential Φ as surfaces along which the vector sum of Newtonian gravity and centrifugal force is zero, so a parcel feels no gravitational acceleration along such surfaces (which are oblate spheroids).

The "effective gravity" is the force acting perpendicular to these surfaces. Newtonian gravity (which always points to Earth's center) is slightly modified by rotation.

$$\vec{g}_{eff} = -\nabla\Phi = \vec{g}_{grav} + \Omega^2 \vec{r}_\perp$$

We reference motion and our coordinate system to Φ surfaces rather than the true sphere. Then $\vec{g}_{eff}$ points locally down everywhere, and we treat is as a constant, $-\nabla\Phi \approx -g\hat{k}$

Momentum equation in rotating frame follows directly from the above transformations:

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

Now we must write the Coriolis force explicitly in the equation.

Rates of change of unit vectors on the sphere In spherical coordinates (unlike Cartesian coordinates) the unit vectors $\hat{i}, \hat{j}, \hat{k}$ change direction with location:

$$\frac{D\hat{i}}{Dt} = \frac{u}{r\cos\theta}\left(\hat{j}\sin\theta - \hat{k}\cos\theta\right)$$
$$\frac{D\hat{j}}{Dt} = -\frac{u\tan\theta}{r}\hat{i} - \frac{v}{r}\hat{k}$$
$$\frac{D\hat{k}}{Dt} = \frac{u}{r}\hat{i} + \frac{v}{r}\hat{j}$$

Momentum equation in spherical coordinates The three components are

$$\frac{Du}{Dt} - \left(2\Omega + \frac{u}{r\cos\theta}\right)(v\sin\theta - w\cos\theta) = -\frac{1}{r\rho\cos\theta}\frac{\partial p}{\partial\lambda}$$
$$\frac{Dv}{Dt} + \frac{wv}{r} + \left(2\Omega + \frac{u}{r\cos\theta}\right)u\sin\theta = -\frac{1}{r\rho}\frac{\partial p}{\partial\theta}$$
$$\frac{Dw}{Dt} - \frac{u^2+v^2}{r} - 2\Omega u\cos\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g$$

Scale analysis for mid-latitude synoptic motion We choose typical scales (orders of magnitude only):

L horizontal length, 1000 km

V horizontal velocity, 10 m s^{-1}

a radius of Earth, 10^7 m

δP horizontal pressure fluctuations, 10 hPa

ρ typical near-surface air density, 1 kg m^{-3}

$T = L/V$ advective time scale, 10^5 s (about 1 day)

Can show that many terms in the full momentum equations are negligible – including most of the metric terms, the vertical component of the Coriolis force, and the contributions to the horizontal Coriolis force associated with the vertical wind.

Primitive equations We make a set of three approximations to the momentum equations simultaneously:

Shallow fluid $r \approx a$ where a is the Earth's radius

Hydrostatic Drop all terms in vertical momentum equation except PGF and gravity

Traditional Drop all terms involving w from the horizontal momentum equation

to get

$$\begin{aligned}
\frac{Du}{Dt} - fv - \frac{uv\tan\theta}{a} &= -\frac{1}{a\rho\cos\theta}\frac{\partial p}{\partial\lambda} \\
\frac{Dv}{Dt} + fu + \frac{u^2\tan\theta}{a} &= -\frac{1}{a\rho}\frac{\partial p}{\partial\theta} \\
0 &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g
\end{aligned}$$

where $f = 2\Omega\sin\theta$ is the projection of the rotation vector $\vec{\Omega}$ onto the local vertical direction.

In vector notation we write $\vec{f} = 2\Omega\sin\theta\hat{k}$

f-plane approximation For motions with small meridional extent we can approximate

$$\vec{f} \approx= \vec{f_0} = f_0\hat{k}$$

where f_0 is evaluated at some reference latitude θ_0.

Then the horizontal momentum equations become

$$\frac{D\vec{u}}{Dt} + \vec{f}_0 \times \vec{u} = -\frac{1}{\rho}\nabla_z p$$

where $\vec{u} = u\hat{i} + v\hat{j}$ is the horizontal velocity.

Inertial motion Consider 2D flow on an f-plane in the absence of any pressure gradient force. Then we get circular motion of radius U/f_0 and frequency f_0.

Beta plane First correction to the f-plane approximation accounting for changes in f in the north-south direction. We write

$$\vec{f} = (f_0 + \beta y)\,\hat{k}$$

where, from a Taylor series expansion,

$$\beta = \frac{2\Omega\cos\theta}{a}$$

We evaluate β at a reference latitude θ_0 and treat it as a constant in the equations of motion.

Boussinesq approximation

- Expand the density field as $\rho = \rho_0 + \delta\rho(x, y, z, t)$ with a constant background density ρ_0
- Scale the equations for small density variations $\delta\rho << \rho_0$
- Break up total pressure field into hydrostatic part that balances ρ_0, and everything else

The Boussinesq equations are

$$\frac{D\vec{v}}{Dt} + 2\vec{\Omega} \times \vec{v} = -\nabla\phi + b\hat{k}$$

$$\nabla \cdot \vec{v} = 0 \qquad \frac{Db}{Dt} = \dot{b}$$

$$\phi = \frac{\delta p}{\rho_0} \qquad b = -g\frac{\delta\rho}{\rho_0}$$

where, if we make the traditional approximation, we would replace $2\vec{\Omega} \times \vec{v}$ with $\vec{f} \times \vec{v}$ in the momentum equation.

Energy budget Total energy equation for adiabatic Boussinesq system is

$$\frac{\partial}{\partial t}\left(\frac{1}{2}v^2 + b\Phi\right) + \nabla \cdot \left(\vec{v}\left(\frac{1}{2}v^2 + b\Phi + \phi\right)\right) = 0$$

where $\Phi = -z$ is the potential, $b\Phi$ is potential energy.

Just like in the fully compressible fluid, total energy is conserved only over a closed domain with rigid boundaries.

Anelastic approximation Similar to Boussinesq but we allow a vertical variation in the mean density:

- $\rho = \tilde{\rho}(z) + \delta\rho(x, y, z, t)$
- Assume $\delta\rho << \tilde{\rho}$
- A more accurate approximation for compressible atmosphere, since ρ decreases roughly exponentially with height.

Continuity equation is

$$\nabla \cdot \vec{u} + \frac{1}{\tilde{\rho}}\frac{\partial}{\partial z}(\tilde{\rho} w) = 0$$

Momentum equation depends on specific choice of $\tilde{\rho}(z)$. Using constant θ, get the same equation as Boussinesq but the buoyancy is

$$b = g\frac{\delta\theta}{\theta_0} \quad \phi = \frac{\delta p}{\tilde{\rho}(z)}$$

Scaling for hydrostatic balance For unstratified Boussinesq fluid, scale analysis of vertical momentum equation using non-divergence gives

$$\frac{\frac{Dw}{Dt}}{\frac{\partial\phi}{\partial z}} \sim \frac{H^2}{L^2}$$

from which we conclude that hydrostatic balance is a *small aspect ratio approximation*.

Rossby number

$$Ro = \frac{V}{fL}$$

gives a measure of importance of advection relative to Coriolis force in horizontal momentum equation. $Ro << 1$ implies flow is close to *geostrophic balance* between Coriolis force and PGF.

$$Ro_T = \frac{1}{fT} = \frac{T_{inertial}}{T}$$

is the temporal Rossby number. Slow motions tend to be geostrophic. $T_{inertial}$ is about 3 hours at mid-latitudes.

Geostrophic balance If $Ro << 1$ then there is an approximate horizontal force balance

$$\vec{f} \times \vec{u} \approx -\frac{1}{\rho}\nabla_z p$$

We use this balance to define *geostrophic* velocities

$$u_g = -\frac{1}{\rho f}\frac{\partial p}{\partial y} \quad v_g = \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

Some properties of geostrophic flow

- Flow is everywhere parallel to isobars
- Flow is anti-clockwise around low pressure if $f > 0$ (clockwise if $f < 0$)
- Flow is approximately non-divergent in the horizontal

Taylor columns For a *barotropic* fluid that is in both geostrophic and hydrostatic balance, we have the *Taylor-Proudman theorem*:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

The fluid cannot have any vertical shear – the motion is the same at every level. This is referred to as "vertical rigidity".

Pressure coordinates Any variable that has a one-to-one (monotonic) relationship with height z can be used as a vertical coordinate instead of z. This works for pressure if the flow is in hydrostatic balance. In this case the adiabatic primitive equations are

$$\frac{D\vec{u}}{Dt} + \vec{f} \times \vec{u} = -\nabla_p \Phi \qquad \frac{D\theta}{Dt} = 0$$

$$\frac{\partial \Phi}{\partial p} = -\alpha \qquad \nabla_p \vec{u} + \frac{\partial \omega}{\partial p} = 0$$

The density ρ no longer appears explicitly in the momentum equation, which is a considerable mathematical simplification.

Geostrophic balance in pressure coordinates

$$\vec{f} \times \vec{u}_g = -\nabla_p \Phi$$

or in components

$$u_g = -\frac{1}{f}\frac{\partial \Phi}{\partial y} \quad v_g = \frac{1}{f}\frac{\partial \Phi}{\partial x}$$

The flow is parallel to lines of constant geopotential height, and $\nabla_p \cdot \vec{u}_g = 0$ on an f-plane.

Thermal wind relation The generalization of the Taylor-Proudman theorem to a baroclinic fluid. For flow that is in geostrophic and hydrostatic balance,

$$\vec{f} \times \frac{\partial \vec{u}_g}{\partial p} = \frac{R}{p}\nabla_p T$$

or in components

$$\frac{\partial v_g}{\partial p} = -\frac{R}{fp}\frac{\partial T}{\partial x} \quad \frac{\partial u_g}{\partial p} = \frac{R}{fp}\frac{\partial T}{\partial y}$$

The vertical shear of the geostrophic wind is proportional to (quasi-horizontal) temperature gradients on pressure surfaces

If cold air lies to the north and $f > 0$, the wind must become *more westerly with height*.

Note that this is a constraint of the *shear* $\frac{\partial u_g}{\partial p}$, not on the value of u_g itself.