ATM 500: Atmospheric Dynamics
Homework 10
Due Tuesday December 4 2018

1. *You may want to read section 4.1 of the text before starting this question.*

   Consider an axisymmetric vortex on a 2D plane. The natural way to describe such motion is with cylindrical coordinates (r, ϕ, z) where r is the radial distance from the center of the vortex, ϕ is the azimuthal angle, and z is the direction perpendicular to the plane. The velocity components can be written $\vec{v} = (u^r, u^\phi, u^z)$. Suppose that the flow is purely azimuthal and varies only with the radial distance, i.e. $u^z = u^r = 0$ and $u^\phi = u^\phi(r)$.

   The vorticity vector is then strictly vertical:

   $$\vec{\omega} = \nabla \times \vec{v} = \omega^z \hat{k}$$

   where

   $$\omega^z = \frac{1}{r}\frac{\partial}{\partial r}\left(r u^\phi\right)$$

   in the component of the curl in cylindrical coordinates.

   a. Show that the circulation C around a circle of radius R centered on the vortex is simply

   $$C = 2\pi R U$$

   where U is the azimuthal velocity at radius R, i.e. $u^\phi(R) = U$.

   b. What does the circulation C tell you about the velocity field inside the circle? Give a formula for one possible velocity profile $u^\phi(r)$ that gives the correct circulation C. Draw a sketch of the flow (velocity) in this vortex between $r = 0$ and $r = R$.

   c. Now repeat part b. for a *different* velocity profile $u^\phi(r)$ that has exactly the same circulation C. Make sure to draw another sketch to compare this to your first vortex.

   d. Calculate the vorticity $\omega^z(r)$ for your two different cases.

   e. Verify that the area integral of the vorticity is equal to C in both cases.

2. *Comparing different classes of shallow water waves*

   In class we will derive a dispersion relation for quasi-geostrophic (QG) Rossby waves for a layer of shallow water on a β plane. For a small QG perturbation linearized about a motionless basic state, the dispersion relation is

   $$\omega = -\frac{\beta k}{K^2 + 1/L_d^2}$$

   (from Vallis equation 5.185 (1st ed.) or 6.65 (2nd ed.)), where k is the zonal wavenumber and $K^2 = k^2 + l^2$ is a total horizontal wavenumber.

Make a graph of ω vs. k for these waves, in the style of Figure 3.8 from Vallis. On the same graph, also plot the dispersion relation for Poincaré waves on an f plane (i.e. shallow-water waves modified by rotation). Use mid-latitude values for f and β. Convince yourself that at any given spatial scale there is a large scale separation in time between the two waves, and thus that QG theory can be thought of as an asymptotic theory for the low-frequency (slow) component of the flow. (You may find it helpful to use a logarithmic axis to visualize both dispersion relations on the same graph).

Make sure to state clearly any assumptions you make when plotting your graph, and be consistent in the assumptions you make for the two classes of wave.

This is not a strictly fair comparison since one theory involves β and the other doesn't, but don't worry about this here.