ATM 500: Atmospheric Dynamics
Homework 10
Due Tuesday December 3 2019

1. *Vallis Essentials question 5.3*

   In the derivation of the quasi-geostrophic equations, geostrophic balance leads to the lowest-order horizontal velocity being divergence-free – that is, $\nabla \cdot \vec{u}_0 = 0$. It seems this can also be obtained from the mass conservation equation at lowest order. Is this a coincidence? Suppose that the Coriolis parameter varied, and that the momentum equation yielded $\nabla \cdot \vec{u}_0 \neq 0$. Would there be an inconsistency?

2. *Based on question 5.6 from Vallis AOFD, 1st edition*

   Given the baroclinic dispersion relation

   $$\omega = \frac{-\beta k}{k^2 + 1/L_d^2}$$

   for what value of k is the zonal component of the group velocity the largest (i.e. the most positive), and what is the corresponding value of the group velocity?

   Estimate the numerical values of the corresponding wavelength and group velocity for the mid-latitude atmosphere, explaining your reasoning.

3. *Vallis Essentials question 6.1*

   In part (c) of this question you will show that I said something incorrect in lecture! A constant westerly background wind actually changes the Rossby wave dispersion relation beyond a simple Doppler shift, unless we are in the barotropic limit where $L << L_d$.

   Consider the flat-bottomed shallow-water quasi-geostrophic equations in standard notation,

   $$\frac{D}{Dt}\left(\zeta + \beta y - \frac{f_0 \eta}{H}\right) = 0$$

   a. How is ζ related to η? Express u, v, η, and ζ in terms of a streamfunction.

   b. Linearize the QGPV equation above about a state of rest, and show that the resulting system supports two-dimensional Rossby waves. Discuss the limits in which the wavelength is much shorter or much longer than the deformation radius.

   c. Now linearize about a *geostrophically balanced state* that is translating uniformly eastward. The means that: $u = U + u'$ and $\eta = \bar{\eta}(y) + \eta'$, where $\bar{\eta}(y)$ is in geostrophic balance with U. Obtain an expression for the form of $\bar{\eta}(y)$. Obtain the dispersion relation for Rossby waves in this system. Show that their phase speed is different from that obtained by adding a constant U to the speed of Rossby waves in part (b). The problem is therefore not *Galilean invariant* – why?

4. *Vallis Essentials question 6.2*

   Consider barotropic Rossby waves obeying the dispersion relation

   $$\omega = Uk - \frac{\beta k}{k^2 + l^2}$$

   where U and/or β vary slowly with latitude.

   a. By re-arranging this expression to obtain an expression for the meridional wavenumber, or otherwise, show that the Rossby waves can only propagate in the meridional direction if

   $$0 < U - c < \frac{\beta}{k^2}$$

   where $c = \omega/k$ is the zonal phase speed.

   b. If the waves approach a latitude where $U = c$ (a 'critical latitude'), show that the meridional wavenumber l becomes large but that the group velocity in the y direction becomes small. Show that Rossby waves generated in the midlatitudes are unlikely to propagate into the tropics (where the mean flow is westward). *This is responsible for the phenomenon known as Rossby wave breaking in the subtropical troposphere*

   c. *FOR BONUS POINTS, NOT REQUIRED.* Suppose that the Rossby waves approach a latitude where $U - c = \beta/k^2$. Calculate the x- and y-components of the group velocity in this limit and infer that a wave will turn away from such a latitude.