ATM 500: Atmospheric Dynamics
Homework 8
Due Friday November 15 2019 by 4 pm

Updated Monday November 11 2019

1. **Dispersion relation for Poincaré waves**

   Starting from the shallow water equations on an f-plane with a flat bottom:

$$\frac{D\vec{u}}{Dt} + f_0 \hat{k} \times \vec{u} = -g \nabla_z \eta$$
$$\frac{D\eta}{Dt} + \eta \nabla \cdot \vec{u} = 0$$

   show that the dispersion relation for small propagating wavy perturbations about a state of rest is

$$\omega^2 = f_0^2 + gH \left(k^2 + l^2\right)$$

   where the notation is standard:

   - ω is the frequency
   - k is the wavenumber in the x direction
   - l is the wavenumber is the y direction
   - H is the resting depth of the fluid

   Yes we derived this result in class, but we skipped many steps. Please present a complete and clear derivation.

2. **Energetics of geostrophic adjustment**

   First, please read Section 4.2.1 on the energetics of the shallow water system. The *potential energy* and *kinetic energy* (both per unit mass) are given by

$$PE = \frac{1}{2} g h^2 \qquad\qquad KE = \frac{1}{2} h |\vec{u}|^2$$

   where h is the total fluid depth.

   Now consider the one-dimensional geostrophic adjustment problem described in class and in Section 4.4.3 of Vallis *Essentials*, where the initial condition is a motionless fluid with a discontinuity of the free surface at $x = 0$.

   a. What is the kinetic energy of the initial state? *(not a trick question)*

   b. Show that in the adjustment to the final geostrophically balanced state, the total kinetic energy increases. More specifically, show that in the final state

$$\int_{-\infty}^{\infty} (KE) dx = + \frac{g \eta_0^2 L_d}{2}$$

c. Starting from the initial state, suppose you could instantly ("magically") rearrange the fluid so that the free surface was flat everywhere with depth H. Would the total potential energy be larger or smaller than the initial state? Explain.

   Here by "total" I mean the integral over the whole domain $\int_{-\infty}^{\infty} dx$.

d. Now consider the actual change in potential energy during the geostrophic adjustment. Show that the potential energy of the final geostrophic state is *smaller* than the initial state. Furthermore, show that this decrease is actually *greater* than the increase in kinetic energy you found above. Thus argue that total energy is reduced by the geostrophic adjustment process.

e. Explain how to reconcile this result with the conservation of total energy in shallow water.

f. Once you've finished working through this problem, please read Section 4.5. Don't worry if the mathematics is unfamiliar, but try to follow the main arguments. Think about the conclusion reached on page 80: *Geostrophic balance is the minimum energy state for a given field of potential vorticity.*

   (This is just a reading assignment, I don't want you to answer any questions or show any work here).