ATM 500: Atmospheric Dynamics
Homework 7

Due Wednesday November 3 2021

1. Consider a compressible ideal gas atmosphere. The environment is characterized by vertical profiles of density and potential temperature $\tilde{\rho}(z), \tilde{\theta}(z)$. A small parcel of fluid is adiabatically displaced upward a small distance δz.

   a. Show that the density difference $\delta\rho$ between the parcel and its new environment is given by
   $$\delta\rho = \frac{\tilde{\rho}}{\tilde{\theta}} \frac{d\tilde{\theta}}{dz} \delta z$$

   b. If the parcel obeys a vertical momentum equation
   $$\frac{Dw}{Dt} = -g \frac{\delta\rho}{\tilde{\rho}}$$
   show that the resulting motion will be vertical oscillations at frequency N where
   $$N^2 = \frac{g}{\tilde{\theta}} \frac{d\tilde{\theta}}{dz}$$
   so long as $\frac{d\tilde{\theta}}{dz} > 0$

2. *Based on Question 3.5 from Vallis*

   Using approximate but realistic values for the observed stratification, what is the *buoyancy period* for (a) the midlatitude troposphere, and (b) the stratosphere. Make sure to cite your data sources and state any assumptions you make.

3. *Based on question 4.1 in Vallis Essentials*

   Using the shallow water equations:

   a. A cylindrical column of air at 30° latitude with radius 100 km expands horizontally (shrinking in depth accordingly) to twice its original radius. If the air is initially at rest, what is the mean tangential velocity at the perimeter after the expansion?

   b. An air column at 60°N with zero relative vorticity ($\zeta = 0$) reaches from the surface to the tropopause, which we assume is a rigid lid, at 10 km. The air column moves zonally onto an area of elevated topography 2.5 km high (this might represent the Tibetan Plateau, for example). What is its relative vorticity? Suppose it then moves southwards to 30°N, staying on the plateau. What is its relative vorticity?

4. **Dispersion relation for Poincaré waves**

   Starting from the shallow water equations on an f-plane with a flat bottom:
   $$\frac{D\vec{u}}{Dt} + f_0 \hat{k} \times \vec{u} = -g \nabla_z \eta$$
   $$\frac{D\eta}{Dt} + \eta \nabla \cdot \vec{u} = 0$$

show that the dispersion relation for small propagating wavy perturbations about a state of rest is

$$\omega^2 = f_0^2 + gH\left(k^2 + l^2\right)$$

where the notation is standard:

- ω is the frequency
- k is the wavenumber in the x direction
- l is the wavenumber is the y direction
- H is the resting depth of the fluid

Yes we derived this result in class, but we skipped many steps. Please present a complete and clear derivation.

5. Read section 4.3.3 of Vallis *Essentials* (or Vallis *AOFD* section 3.8.3 (2nd edition) or 3.7.3 (1st edition)). The *Kelvin wave* is a special kind of gravity wave that exist in the presence of a lateral boundary. The wave is in geostrophic balance in the direction perpendicular to the boundary, but unbalanced (and propagating at phase speed $\sqrt{gH}$) in the direction along the boundary. (These waves are particularly important for our understanding of tropical phenomena such as El Niño).

   Why did we not find the Kelvin wave when we did our standard wave analysis of the rotating shallow water equations in class?