# ATM 500: Atmospheric Dynamics
# Homework 9

**Due Wednesday November 17 2021**

*Typo corrected 11/13/2021*

1. *Vallis Essentials question 5.3*

   In the derivation of the quasi-geostrophic equations, geostrophic balance leads to the lowest-order horizontal velocity being divergence-free – that is, $\nabla \cdot \vec{u}_0 = 0$. It seems this can also be obtained from the mass conservation equation at lowest order. Is this a coincidence? Suppose that the Coriolis parameter varied, and that the momentum equation yielded $\nabla \cdot \vec{u}_0 \neq 0$. Would there be an inconsistency?

2. An important motivation for deriving the quasi-geostrophic equations was to filter out fast gravity waves from the equations of motion. Verify that we achieved this! Specifically, start from the shallow-water quasi-geostrophic potential vorticity conservation equation

   $$\frac{D_g}{Dt}\left[\nabla^2\psi + \beta y - \frac{\psi}{L_d^2}\right] = 0$$

   (where $\psi$ is the geostrophic streamfunction as defined in class). First, make the $f$-plane approximation. Then linearize for small perturbations away from a state of rest and show that there are no propagating wave solutions.

   You can show this by following our standard linear wave analysis technique and concluding that the only possible solution for the frequency is $\omega = 0$.

   *Hint: what is the value of $\beta$ on the $f$-plane?*

3. Now *on the beta plane*, show that small perturbations away from a state of rest must obey the dispersion relation

   $$\omega = \frac{-\beta k}{K^2 + 1/L_d^2}$$

   where $k$ is the zonal ($x$ direction) wavenumber and $K^2 = k^2 + l^2$ is a total horizontal wavenumber.

   (These are shallow-water Rossby waves, whose properties we will be discussing).

   *Hint: an important term in the advection of QGPV is $\beta v$, where $v = \frac{\partial \psi}{\partial x}$ is the meridional component of (geostrophic) velocity.*

4. *Comparing different classes of shallow water waves*

   Make a graph of $\omega$ vs. $k$ for the waves discussed in question 3, in the style of Figure 4.4 from Vallis *Essentials*. Use the sign convention that $\omega >= 0$ while $k < 0$ corresponds to a westward-propagating wave. On the same graph, also plot the dispersion relation for Poincaré waves on an $f$ plane (i.e. shallow-water waves modified by rotation). Use mid-latitude values for $f$ and $\beta$. (You may find it helpful to use a logarithmic axis to visualize both dispersion relations on the same graph).

Use your graph to argue that at any given spatial scale there is a large scale separation in time between the two waves. Thus convince yourself that QG dynamics can be thought of as an asymptotic theory for the low-frequency (slow) component of the flow.

Make sure to state clearly any assumptions you make when plotting your graph, and be consistent in the assumptions you make for the two classes of wave.

*This is not a strictly fair comparison since one theory accounts for $\beta$ and the other doesn't, but don't worry about this here.*