ATM 500: Atmospheric Dynamics
Homework 8
Due Wednesday November 10 2021

1. **Energetics of geostrophic adjustment**

   First, please read Section 4.2.1 on the energetics of the shallow water system. The *potential energy* and *kinetic energy* (both per unit mass) are given by

   $$PE = \frac{1}{2} g h^2 \qquad\qquad KE = \frac{1}{2} h |\vec{u}|^2$$

   where h is the total fluid depth.

   Now consider the one-dimensional geostrophic adjustment problem described in class and in Section 4.4.3 of Vallis *Essentials*, where the initial condition is a motionless fluid with a discontinuity of the free surface at $x = 0$.

   a. What is the kinetic energy of the initial state? *(not a trick question)*

   b. Show that in the adjustment to the final geostrophically balanced state, the total kinetic energy increases. More specifically, show that in the final state

   $$\int_{-\infty}^{\infty} (KE) dx = + \frac{g \eta_0^2 L_d}{2}$$

   c. Starting from the initial state, suppose you could instantly ("magically") rearrange the fluid so that the free surface was flat everywhere with depth H. Would the total potential energy be larger or smaller than the initial state? Explain.

   Here by "total" I mean the integral over the whole domain $\int_{-\infty}^{\infty} dx$.

   d. Now consider the actual change in potential energy during the geostrophic adjustment. Show that the potential energy of the final geostrophic state is *smaller* than the initial state. Furthermore, show that this decrease is actually *greater* than the increase in kinetic energy you found above. Thus argue that total energy is reduced by the geostrophic adjustment process.

   e. Explain how to reconcile this result with conservation of total energy in shallow water.

   f. Once you've finished working through this problem, please read Section 4.5. Don't worry if the mathematics is unfamiliar, but try to follow the main arguments. Think about the conclusion reached on page 80: *Geostrophic balance is the minimum energy state for a given field of potential vorticity.*

   (This is just a reading assignment, I don't want you to answer any questions or show any work here).

2. *Based on question 4.6 in Vallis Essentials*

   In this question you will investigate a variant of the geostrophic adjustment problem using the shallow water equations. Here the initial imbalance exists because of a discontinuity in the velocity field rather than in the free surface or pressure field.

   a. Starting from the conservation of potential vorticity $Q = (f + \zeta)/h$ for parcels in the shallow water system, derive the linear result

   $$\frac{\partial q}{dt} = 0, \qquad q = \zeta' - f_0 \frac{\eta'}{H}$$

   for small perturbations of the height and velocity field on an f-plane with a flat bottom. *(We used this result in class already but did not show the derivation)*

   b. Suppose we have an infinite slab of shallow water on an f-plane. Initially, the fluid surface is flat, the zonal velocity u is zero everywhere, but there is a meridional velocity given by

   $$v(x) = v_0 \,\mathrm{sgn}(x)$$

   where v_0 is a constant. This means that the fluid is moving northward at a speed v_0 everywhere to the east of $x = 0$, and southward at speed v_0 everywhere to the west of $x = 0$.

   Verify that this initial condition is NOT a steady solution of the shallow water equations (i.e. show that there must be a non-zero time tendency of velocities or surface height or both, at least somewhere in the domain).

   c. Show that $q = 0$ everywhere in the domain *except* at $x = 0$.

   d. After the adjustment the system will reach a final steady state that can be described with a geostrophic streamfunction $\psi(x, y)$ satisfying

   $$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}$$

   Since the linearized PV field is fixed in space, the final steady state is the solution of this equation:

   $$\frac{d^2\psi}{dx^2} - \frac{\psi}{L_d^2} = q$$

   where q is the initial distribution of PV that you worked out above.

   Solve the differential equation for ψ separately for $x > 0$ and $x < 0$. Apply a physically sensible boundary condition for $x \to \pm\infty$. Be sure to explain your reasoning. There should be one unknown constant left in each solution.

   e. Because of the discontinuity in the initial velocity field at $x = 0$, the initial relative vorticity ζ is locally infinite at $x = 0$, and therefore $q(x)$ is also. We can describe this with a so-called 'delta function'

   $$q(x) = 2v_0\delta(x)$$

where $\delta(x) = 0$ by definition everywhere except $x = 0$, but it also has the property that if we integrate over any interval, the result is finite and independent of the interval:

$$\int_{x_1}^{x_2} \delta(x) dx = 1$$

so long as the interval contains $x = 0$ (i.e. $x_1 < 0$ and $x_2 > 0$).

To evaluate your unknown constants and complete your solution, apply these two boundary conditions:

- η' is continuous at the boundary in the final state.
- The integral $\int_{-\infty}^{+\infty} q \, dx$ must be the same in the initial and final states.

Using these conditions, what is $\psi(x)$?

f. Solve for the final surface height and velocity fields. Draw a sketch of the final adjusted state. Is there anything unusual or perhaps unphysical about the final state?