MTB15: Extra-Tropical Weather Systems

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Contents

Chapter 1

Phenomenology of Weather Systems

In this course we aim to discuss the meteorology of the weather systems typical of the extra-tropics. The distinction being drawn between tropical and extra-tropical systems is not a clear-cut one in the sense that certain phenomena, such as certain types of convective storms, are similar in the tropics and extra-tropics. One aspect to note though is that we intend to include aspects of the meteorology on a range of horizontal scales from small-scale through mesoscale to synoptic scale. We do not include the microscale such as is involved in boundary layer and micrometeorology nor do we include the global scale such as discussed in the global circulation. The focus is on general dynamical concepts that attempt to explain observed behaviour of the weather systems. Often implied will be a knowledge of the typical structure of these weather systems - descriptions of this can be found in some of the textbooks recommended for this course.

One of the key differences between the tropics and extra-tropical systems is apparent when we look at the synoptic scale aspects. In the extra-tropics the role of the Coriolis force is crucial in explaining the synoptic scale features whereas in the tropics it often, but not always, plays a less important role. For example many aspects of tropical cyclones can be understood using a zero Coriolis parameter whereas for extra-tropical cyclones geostrophic balance is fundamental.

A list of the phenomena which come under the umbrella of this course are: troughs, ridges, anticyclones, cyclones, polar lows, fronts, rainbands, convection, and orographic flows. In this section we give a few salient features of these systems, more details should be found from the recommended texts.

1.1 Troughs and ridges

At upper levels the mid-latitude zones are dominated by regular trough-ridge patterns. These are typically several thousand kilometres in horizontal extent and often have their amplitude maximized on the tropopause. They do however extend significantly throughout the troposphere. The origin of these waves is to be found in the global circulation and is not the aim of our discussion here. However it soon becomes apparent looking at satellite images that these trough-ridge systems can be associated with development of cloud systems and indeed with surface features such as cyclones. This implies that they may have a role in generating vertical motion in the troposphere. For example it is seen that ahead (that is downstream) of a trough, and consequently upstream of a ridge, there is often a region of ascent. We will be giving a theory for the generation of such vertical motion.

Another way to describe these patterns is in terms of jet streaks. These are clearly defined on say 250 mb geopotential height charts as zones of large upper-level flow. It is often more convenient to talk about jet stream entrances and exits implied by these jet streaks than to discuss the associated trough-ridge pattern. But these are different ways to describe the same phenomena.

The focus on the tropopause is one which will recur frequently in this course. The extra-tropical tropopause is an imaginary boundary between the troposphere and stratosphere which can be thought of as a two-dimensional flexible membrane. In climatological terms this membrane is oriented in the zonal direction and it slopes strongly from equator to pole. However from day-to-day it is much more contorted than that picture implies and exhibits strong meridional variations associated with the trough-ridge patterns. It can, for example, become folded such that over a given location a vertical sounding encounters the troposphere both between the ground and the stratosphere but also higher up in a layer which is apparently sandwiched within the stratosphere.

1.2 Cyclones and anticyclones

Probably the most common weather phenomenon in the extra-tropics on the synoptic scale is the extra-tropical cyclone. These are particularly evident on surface weather charts and they exist in what are predominantly oceanic storm-tracks. The reasons for the existence of these storm-tracks is a question addressed in studies of the global circulation. As an example the north Atlantic storm-track stretches from the east coast of the USA/Canada and ends in north-west Europe. At any one time this region is occupied by a set of low pressure centres with, of course, highs or anticyclones in between. These lows travel from west to east and have a well defined life-cyle of growth, maturity and decay. A typical timescale for this life-cycle is between 1 and 3 days. A typical horizontal length scale for an individual cyclone is between 1000 and 2000 km. They are perhaps the most studied of all meteorological phenomena. The cyclone has embedded within it a variety of mesoscale sub-structures which we will discuss later. Although particularly evident on surface weather charts cyclones often, but not always, extend throughout the depth of the troposphere. In this course we will be describing the basic dynamical processes which account for the growth and evolution of extra-tropical cyclones.

1.3 Polar lows

Particularly noticeable phenomena occuring in the high latitudes are called polar lows and these occur well to the north of the main oceanic storm-track. Thus they are essentially cold (polar) air-mass features. They have the appearance of smaller-scale extra-tropical cyclones but sometimes they take on more of the character of a tropical cyclone such as exhibiting an eye. It is clear that air-sea surface fluxes are very large in association with polar lows as cold air over the polar ice-sheets moves south over relatively warm sea-water. This may be why they can take on characteristics more commonly associated with tropical cyclones. They can affect the UK in northerly flow outbreaks and indeed have led to some of the largest snowfalls experienced in northern Britain. Also in the cold air-mass there can be so-called comma cloud disturbances. These take their name from the characteristic cloud patterns they exhibit and they can develop into fully fledged cyclones of small scale. Their dynamics remains somewhat unknown at this stage.

1.4 Fronts

Within extra-tropical cyclones there are zones where horizontal temperature gradients are increased bringing cold and warm air-masses into close proximity. These are called fronts; a name given by the Bergen School in the 1920's drawing a parallel between the two sides fighting in the First World War and the two "warring" air-masses. The Norwegians provided us with what is still the way in which fronts are represented on weather charts. Much of the weather associated with extra-tropical cyclones is focussed at these fronts; for example, precipitation and wind-shifts. It is therefore obvious that there must be vertical motion at active fronts. The process of intensification of a front is referred to as frontogenesis whilst its decay is called frontolysis.

There are many different distinct classes of front: cold, warm, occluded, split, ana, kata, upper and surface. In this course we will be discussing the dynamical processes responsible for frontogenesis and providing a theory for the vertical motion. Fronts are often quasi two-dimensional being perhaps a thousand kilometres long. Their across-front width can be extremely narrow but certainly of order 100 km or less.

1.5 Rainbands

The precipitation at fronts is usually organised into a set of bands lying parallel to the front but with a cross-front width of order 50 km. This organisation is important if accurate forecasts of rainfall associated with fronts are to be given. The ascent in these rainbands rarely exceeds 20 cms^{-1} leading to rainfall rates of order 10 mmh^{-1}. These phenomena have recently been discovered to be the result of processes distinct from those associated with convection.

1.6 Convection

In this course we will concentrate on severe convective storms or cumulonimbus that persist for several hours. These contribute significantly to local weather and are distinct from short-lived non-precipitating clouds. Such severe storms can be classified into a small set of distinct categories: squall lines, multicellular storms and supercells. The typical horizontal scale of such storms is of order 50 km so that they are phenomena which do not rely on the Coriolis force governing their dynamics. Convection can occur if the vertical lapse rate is such that with a small amount of local heating there can be unstable upward parcel displacements. The thermodynamics of this process are easily seen on tephigrams taken from soundings made just prior to convection breaking out. However the organisation of the storm relies on the wind profile in which these storms exist. The storms are composed of distinct movements of air called updraughts and downdraughts. Their longevity relies on a positive feedback between these two draughts. Such convective storms are inherently non-linear as the updraught is heated via latent heat release as cloud forms whereas the descent is partly saturated but also partly unsaturated.

Severe convective storms can have intense updraughts with ascent of order 20 ms^{-1}. This leads to locally large rainfall rates (eg: $100mmh^{-1}$) and local accumulations of rainfall. Associated with some storms are tornadoes which are intense local vortices which can cause extreme local damage. Tornadogenesis is a process which is the subject of current research and will be discussed in this course.

1.7 Orographic flows

Mountains can cause a significant effect on the atmosphere on a range of scales. Here we will focus on the small scale and mesoscale aspects of mountain meteorology. Depending on aspects of the flow impinging on mountains there are a variety of local perturbations to the flow caused by that mountain. These include: gravity waves, lee vortices, blocked flow, trapped lee waves, lenticular clouds and lee cyclogenesis. Also the mountains act as a drag on the atmosphere and this is thought to be an important source of deceleration of the large-scale flow of the atmosphere. The drag is of two kinds: gravity wave and surface pressure drag. Valleys can also generate distinctive flow regimes.

Another important local flow regime is called the sea/land breeze and this relies on the differential heating of the land relative to the sea at the coast. It is one of several coastal meteorological phenomenon which makes the coastal margins of special and current scientific interest. Often the term “topographic” is used to encompass both orographic and coastal forcings.

Chapter 2

Vorticity & Potential Vorticity Concepts

2.1 Boussinesq approximation

In this course we will use the equations with height as the vertical coordinate. The purpose of the Boussinesq approximation is to simplify the equations of motion by removing the large static state of the atmosphere to leave the interesting remainder which is responsible for the motion and weather systems. We define a basic state which is only dependent on height, and a small deviation:

$$\theta(x,y,z,t) = \overline{\theta}(z) + \theta'(x,y,z,t) \tag{2.1}$$
$$p(x,y,z,t) = \overline{p}(z) + p'(x,y,z,t) \tag{2.2}$$
$$\rho(x,y,z,t) = \overline{\rho}(z) + \rho'(x,y,z,t) \tag{2.3}$$

The definition of the basic state is:

$$\frac{d\overline{p}}{dz} = -\overline{\rho}g \tag{2.4}$$
$$\overline{\rho} = \rho_0 e^{-\frac{z}{H_0}} \tag{2.5}$$

where H_0 is the density scale height. We can substitute equations (2.1), (2.2) and (2.3) into the definition of potential temperature, $\theta = T\left(\frac{p}{p_0}\right)^{-\frac{R}{c_p}}$ to give a relationship between the deviation quantities:

$$\frac{\theta'}{\overline{\theta}} = \frac{p'}{\gamma\overline{p}} - \frac{\rho'}{\overline{\rho}} \tag{2.6}$$

where $\gamma = \frac{c_p}{c_v}$. We now substitute equations (2.1), (2.2) and (2.3) into the momentum equations. This is straightforward except for the vertical momentum equation. Consider the term $\frac{1}{\rho}\frac{\partial p}{\partial z}+g$. This can be approximated in the following way, using a Taylor expansion of $\frac{1}{(\overline{\rho}+\rho')}$:

$$\frac{1}{\rho}\frac{\partial p}{\partial z}+g=g\frac{\rho'}{\overline{\rho}}+\frac{1}{\overline{\rho}}\frac{\partial p'}{\partial z}+O(p'\rho') \tag{2.7}$$

We can neglect the term of order deviation quantities squared. Using the definition of the scale height $H_0=\frac{R\overline{T}}{g}$, the gas law $p=\rho RT$, equation (2.6) and some (!) algebra we can obtain:

$$\frac{1}{\rho}\frac{\partial p}{\partial z}+g=-g\frac{\theta'}{\overline{\theta}}+\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)-\frac{N^2p'}{g\overline{\rho}} \tag{2.8}$$

where $N^2=\frac{g}{\overline{\theta}}\frac{d\overline{\theta}}{dz}$ is the static stability parameter. It is a good approximation to take $\frac{\partial}{\partial z}\gg\frac{N^2}{g}$ and if we define $\phi'=\frac{p'}{\overline{\rho}}$ (which is a normalised pressure perturbation with the same dimensions as a geopotential) then the vertical momentum equation becomes:

$$\frac{Dw}{Dt}+\frac{\partial\phi'}{\partial z}=g\frac{\theta'}{\overline{\theta}} \tag{2.9}$$

In practice it is also a good approximation to take $\overline{\theta}\approx\theta_0$ in this equation, where θ_0 is a constant. Also we can define a buoyancy parameter $b'=g\frac{\theta'}{\theta_0}$. So for synoptic scale motion where the vertical acceleration can be neglected we get the Boussinesq hydrostatic equation:

$$\frac{\partial\phi'}{\partial z}=b' \tag{2.10}$$

This can be compared to the hydrostatic relation in pressure coordinates which is:

$$\frac{\partial\phi}{\partial p}=-\frac{R}{p}\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta \tag{2.11}$$

The horizontal momentum equations are given by:

$$\frac{D\underline{v}}{Dt}+fk\times\underline{v}+\nabla\phi'=0 \tag{2.12}$$

where $\underline{v}$ is the horizontal velocity vector.

Also using the split of θ into a mean and deviation parts the thermodynamic equation $\frac{D\theta}{Dt}=0$ is easily shown to be:

$$\frac{Db'}{Dt}+wN^2=0 \tag{2.13}$$

The mass conservation equation is:

$$\frac{D\rho}{Dt}+\rho\nabla.\underline{u}=0 \tag{2.14}$$

The alternative form of this equation is obtained by dividing the substantial derivative into a local rate of change and an advective component:

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \underline{u}) = 0 \tag{2.15}$$

Substituting in the split for the density into a basic state part and a deviation we obtain:

$$\frac{\partial \rho'}{\partial t} + \nabla . (\overline{\rho} \underline{u}) = 0 \tag{2.16}$$

The anelastic approximation, whereby we neglect the local time rate of change of density which produces sound waves only, is a very good approximation giving a very simple continuity equation:

$$\nabla . (\overline{\rho} \underline{u}) = 0 \tag{2.17}$$

Another way to write this is obtained by remembering that $\overline{\rho}$ is only a function of height is:

$$\nabla . \underline{v} + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} (\overline{\rho} w) = 0 \tag{2.18}$$

2.2 Quasi-geostrophic theory

2.2.1 Resume of geostrophic definitions

The definitions of geostrophic quantities are:

geostrophic derivative:

$$D_g = \frac{\partial}{\partial t} + \underline{v}_g . \nabla \tag{2.19}$$

geostrophic streamfunction:

$$\psi = \frac{p'}{\rho_0 f_0} \tag{2.20}$$

buoyancy and hydrostatic relation:

$$b' = \frac{g}{\theta_0} \theta' = f_0 \frac{\partial \psi}{\partial z} \tag{2.21}$$

geostrophic wind:

$$(u_g, v_g) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x} \right) \tag{2.22}$$

relative vorticity:

$$\xi_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = {\nabla_h}^2 \psi \tag{2.23}$$

thermal wind relationships:

$$\left(\frac{\partial u_g}{\partial z}, \frac{\partial v_g}{\partial z}\right) = \frac{1}{f_0}\left(-\frac{\partial b'}{\partial y}, \frac{\partial b'}{\partial x}\right) \tag{2.24}$$

$$\frac{\partial \xi_g}{\partial z} = \frac{1}{f_0}{\nabla_h}^2 b' \tag{2.25}$$

Other quantities are as follows: p' is the deviation pressure from a hydrostatic reference state, ρ_0 is the constant density, f_0 is a constant value of the Coriolis parameter.

2.2.2 QG vorticity and thermodynamic equations

The quasi-geostrophic (QG) equations in height coordinates consist of the vorticity equation and the thermodynamic equation. As we are particularly interested in the generation of circulation systems such as highs and lows we focus on the vertical component of vorticity. Also we use a scaling argument to consider only those horizontal scales typical of synoptic systems. This means that the Rossby number is assumed to be small. In this case the full three-dimensional vector vorticity equation can be shown by scale analysis to be dominated by the following form for the vertical component of vorticity:

$$\frac{D}{Dt}(f+\xi) = -(f+\xi)\nabla.\underline{v} \tag{2.26}$$

The continuity equation for the Boussinesq system can then be used to obtain:

$$\frac{D}{Dt}(f+\xi) = \frac{(f+\xi)}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}w\right) \tag{2.27}$$

It is often true on scale grounds that for synoptic systems: $f \gg \xi$ and also that the winds for advection can be approximated by the geostrophic winds so that the QG vorticity equation can be written as:

$$D_g\left(f+\xi_g\right) = \frac{f}{\overline{\rho}}\frac{\partial}{\partial z}\left(\overline{\rho}w\right) \tag{2.28}$$

where $D_g = \frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}$ is the so-called geostrophic derivative. In this course for mathematical simplicity we will assume that the atmosphere has constant mean density; it is straightforward to include an exponential variation of density with height.

Using equation (2.13) and using the geostrophic advection we obtain directly the QG thermodynamic equation:

$$D_g b' = -wN^2 \tag{2.29}$$

2.3 Vorticity thinking

A key consequence of the continuum hypothesis in fluid dynamics is the concept of "action-at-a-distance". This is best exemplified by first considering the vertical component of vorticity, ξ, of a flow which is two-dimensional in the (x, y) plane. This is defined by the equation:

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = {\nabla_h}^2 \psi_s \tag{2.30}$$

where ψ_s is the streamfunction for two-dimensional (non-divergent) flow. Equation (2.30) can be viewed in two complementary ways. The first is as a way of calculating the vorticity if we know the winds or stream function. The second is as a way to calculate the winds or streamfunction if we know the vorticity distribution everywhere. This second way is conceptually useful because in two-dimensional flow the vorticity is conserved following the motion and so if the vorticity distribution is known at a given time it can be found, in principle, at subsequent times. Knowing the vorticity then equation (2.30) can be solved to find the flow. This is reminiscent of the theory of electrostatics where elements of charge (vorticity in our case) are said to "induce" an electric field (streamfunction in our case). Here we will often refer to the vorticity inducing flow or alternatively as the flow being attributed to the vorticity source.

The basic building blocks of atmospheric dynamics then become elements or particles of vorticity. The key point that we know from electrostatics is that localised charges lead to a electric field which is not localised but rather global in nature. Hence anomalies of vorticity induce flow both inside and outside the anomaly. Therefore we can speak of action-at-a-distance. The idea of vorticity being the centre of action was introduced by Rossby in the 1940's but the connection between fluid dynamics and electromagnetism dates back to the turn of the century and the work of Vilhelm Bjerknes.

2.4 Potential Vorticity Thinking

It is possible, and necessary, to extend the two-dimensional flow ideas described by vorticity thinking to a fully three-dimensional stratified atmosphere. This extension involves the quantity known as potential vorticity which is in addition conserved following the motion.

2.4.1 Conservation of QG Potential Vorticity

The conservation of quasi-geostrophic potential vorticity follows from the QG vorticity and thermodynamic equations. Substitute w from equation (2.29) into the vorticity equation (2.28):

$$D_g(f + \xi_g) = f_0 \frac{\partial}{\partial z} \left(-\frac{1}{N^2} D_g b' \right) \tag{2.31}$$

A key approximation is now made which is that the static stability parameter N^2 is only a function of height, i.e. it cannot be a function of the horizontal coordinates. Then equation (2.31) can be written as:

$$D_g(f + \xi_g) = -f_0 \frac{\partial}{\partial z} \left(D_g \left(\frac{b'}{N^2} \right) \right) \tag{2.32}$$

It turns out that a consequence of thermal wind balance is that the vertical derivative and the geostrophic derivative can be interchanged. So using thermal wind balance given in equation (2.24) we can show that the derivatives can be written as:

$$\frac{\partial}{\partial z}\left(D_g\left(\frac{b'}{N^2}\right)\right) = D_g\left(\frac{\partial}{\partial z}\left(\frac{b'}{N^2}\right)\right) \tag{2.33}$$

Hence the conservation equation for QG potential vorticity is obtained by using equation (2.33) in equation (2.32):

$$D_g\left(f + \xi_g + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2}b'\right)\right) = 0 \tag{2.34}$$

The quantity inside the brackets is called the QG potential vorticity and it can be written as $q = f + q'$ where q' is the so-called perturbation potential vorticity. This can be written purely in terms of the geostrophic streamfunction ψ:

$$q' = \nabla_h{}^2\psi + \frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial\psi}{\partial z}\right) \tag{2.35}$$

Note that the geostrophic streamfunction is the extension to a fully three-dimensional flow of the two-dimensional streamfunction discussed earlier. Unlike its two-dimensional counterpart the vertical derivative of the geostrophic streamfunction has a meteorological significance in that it gives the temperature field.

Consequently the potential vorticity equation (2.34) can also be written purely in terms of $\psi(x, y, z, t)$. It is an equation which describes the time development of potential vorticity and/or streamfunction. To integrate this equation forward in time boundary conditions are needed and these can be provided from the thermodynamic equation (2.29). For example the system can be solved between two bounding rigid surfaces at which $w = 0$ or:

$$w = -\frac{D_g(\frac{\partial\psi}{\partial z})}{N^2} = 0 \tag{2.36}$$

This method of solution is used for example in the Eady problem for baroclinic instability.

2.4.2 Invertibility

The QG potential vorticity definition equation (2.35) can be viewed as an inversion equation whereby if the potential vorticity were to be known then the equation can be inverted to find the geostrophic streamfunction associated with that potential vorticity anomaly. To perform that inversion it is necessary to know boundary conditions on ψ such as the buoyancy on horizontal bounding surfaces. This is the counterpart to the action-at-a-distance idea that was described earlier for two-dimensional flow. That concept has been expanded to three-dimensions as long the basic geostrophic and hydrostatic balances apply. The potential vorticity then takes on the role that vorticity did in two-dimensional flow of being the centre of action. Anomalies of potential vorticity can be thought of as inducing flow and temperature or alternatively we can say that aspects of the flow and temperature can be attributed to individual potential vorticity anomalies.

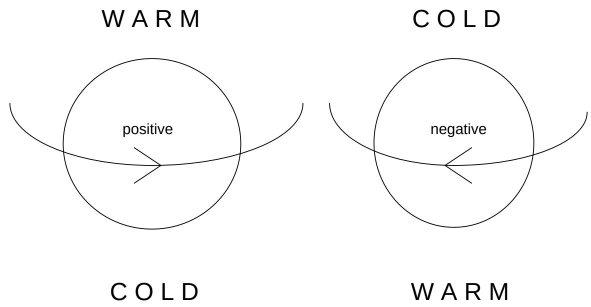

Figure 2.1: A schematic of single ball anomalies of potential vorticity in the $x - z$ plane with their associated circulation and temperature anomalies.

An example of this invertibility idea, as it has become to be known, is to consider a single "ball" of uniform QG potential vorticity which is embedded in a flow with uniform static stability and Coriolis parameter. Define a radial type coordinate by $r = \sqrt{x^2 + y^2 + \frac{N^2}{f^2} z^2}$ and suppose the ball of potential vorticity is defined by $q' = \hat{q}$ for $r < r_0$ and $q' = 0$ otherwise. Note that the origin of the coordinate system is chosen to be at the centre of the ball. Then it can be shown that the solution of equation (2.35) is given by:

$$\psi = -\frac{{r_0}^3}{3r}\hat{q} \quad \text{if } r > r_0 \tag{2.37}$$

$$\psi = -\left(\frac{{r_0}^2}{2} - \frac{r^2}{6}\right)\hat{q} \quad \text{if } r < r_0 \tag{2.38}$$

The flow and potential temperature attributable to the QG potential vorticity anomaly is calculated via the geostrophic and hydrostatic equations using the differentiation rules:

$$\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{x}{r}, \frac{y}{r}, \frac{N^2 z}{f^2 r}\right)\frac{\partial}{\partial r} \tag{2.39}$$

A relatively simple rule which follows from this solution is that a positive q' anomaly induces cyclonic flow and has warm air above and cold air below the anomaly whilst a negative q' anomaly has anticyclonic flow and has cold air above it and warm air beneath it; shown schematically in Figure (2.1). Consequently within an anomaly the perturbation static stability is of the same sign as the potential vorticity of the anomaly. In Figure (2.2) the total θ and v distributions are shown and the thermal and wind properties attributable to the ball are apparent. It is surprising, but it follows from the simple form of the elliptic equation for QG potential vorticity, that these rules are obeyed for nearly all anomalies no matter what their shape.

As will be discussed in the next section once ψ is known then the geostrophic flow and temperature can be found and then $\underline{Q}$ can be deduced. Hence the omega equation forcing is known and if the omega equation is solved we can obtain the vertical motion attributable

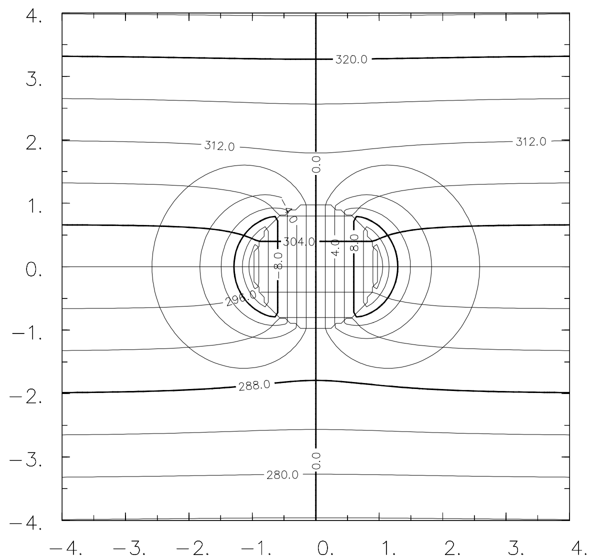

Figure 2.2: A vertical cross-section of θ and v (wind component into the page) in the plane $y = 0$ for a uniform QG potential vorticity ball of dimension $r_0 = 200km$ and magnitude $\hat{q} = 2f$. The horizontal axis is scaled by r_0 and the vertical axis is scaled by $\frac{N}{f} r_0$.

to a given potential vorticity feature. Notice that there is no time integration involved here and so inversion gives the synoptic structure of the atmosphere.

2.4.3 Ertel-Rossby Potential Vorticity

In fact the concept of potential vorticity is even more general than that implied by the quasi-geostrophic theory. Starting from the primitive (non-hydrostatic) equations it is possible to show that there is a quantity, PV, which is conserved following the motion if the flow is frictionless and adiabatic:

$$\frac{D}{Dt} PV = 0; PV = \frac{1}{\rho} \underline{\zeta}.\nabla\theta \tag{2.40}$$

Here $\underline{\zeta}$ is the three-dimensional vorticity vector. The conservation of PV was first proven in mathematical generality by Ertel 1942 but Rossby discusses its relationship to vorticity at this time also. The quantity PV is now commonly known as the Ertel-Rossby potential vorticity even though it does not have the units of vorticity! This confusing aspect can be overcome by scaling PV with a constant static stability and density to form a quantity with the appropriate units: $(\frac{\rho_0 g}{{N_0}^2 \theta_0}) PV$.

(The relationship between the PV and the quasi-geostrophic potential vorticity, q, can be given by linearising the expression above for the PV about a state with a reference static stability and a flow with no basic state vorticity.)

A picture of what this conservation principle means can be obtained by imagining a vortex tube attached top and bottom to different θ surfaces. If the θ surfaces move further apart the ends of the vortex tube remain attached to the θ surfaces and so the tube is streched and thinned and thus will spin cyclonically to conserve angular momentum.

If the flow is frictional and diabatic then the PV is no longer conserved but satisfies the evolution equation:

$$\frac{DPV}{Dt} = \frac{1}{\rho}\left(\underline{\zeta}.\nabla\frac{D\theta}{Dt} + \nabla \times \underline{F}.\nabla\theta\right) \tag{2.41}$$

where $\underline{F}$ is the frictional force vector.

So friction and diabatic heating can produce local anomalies of potential vorticity in the flow. For example if there is diabatic heating of the atmosphere due to latent heat release in a cloud then the PV increases below and decreases above the heating maximum.

Chapter 3

Vertical Motion

3.1 Omega equation: VA and TA form

The omega equation arises by combining the vorticity and thermodynamic equations so as to remove the time derivatives. The resulting equation is then an equation for the vertical motion which is diagnostic as it applies at each instant. We here retain the terminology of the "omega" equation even though we shall be using the vertical velocity; the equation was originally derived in pressure coordinates.

We begin by writing the vorticity and thermodynamic equations in the following way:

$$\frac{\partial}{\partial t}\xi_g + \underline{v}_g.\nabla\xi_g + \beta v_g = f_0\frac{\partial w}{\partial z} \tag{3.1}$$

$$\frac{\partial}{\partial t}b' + \underline{v}_g.\nabla b' = -N^2 w \tag{3.2}$$

The time derivatives can be removed by using a form of the thermal wind relationship:

$$\frac{\partial \xi_g}{\partial z} = \frac{1}{f_0}{\nabla_h}^2 b' \tag{3.3}$$

This equation can be differentiated with respect to time (local time-rate-of-change) and multiplied by f_0 to give:

$$\frac{\partial}{\partial t}\left(f_0\frac{\partial \xi_g}{\partial z}\right) = \frac{\partial}{\partial t}{\nabla_h}^2 b' \tag{3.4}$$

Now take $f_0\frac{\partial}{\partial z}$ of equation (3.1) and subtract ${\nabla_h}^2$ of equation (3.2) to give:

$$\frac{\partial}{\partial t}\left(f_0\frac{\partial \xi_g}{\partial z}\right) - \frac{\partial}{\partial t}{\nabla_h}^2 b' = {\nabla_h}^2(\underline{v}_g.\nabla b') - f_0\frac{\partial}{\partial z}\left(\underline{v}_g.\nabla\xi_g\right) - f_0\beta\frac{\partial v_g}{\partial z} + {f_0}^2\frac{\partial^2 w}{\partial z^2} + N^2{\nabla_h}^2 w \tag{3.5}$$

The time derivatives cancel using equation (3.4) and we then obtain the omega equation using some re-arrangement of equation (3.5):

$$N^2{\nabla_h}^2 w + {f_0}^2 \frac{\partial^2 w}{\partial z^2} = S \tag{3.6}$$

where the term S is the so-called source term and contains only geostrophically determined quantities:

$$S = f_0 \frac{\partial}{\partial z}\left(\underline{v}_g.\nabla\xi_g\right) - {\nabla_h}^2(\underline{v}_g.\nabla b') + f_0\beta\frac{\partial v_g}{\partial z} \tag{3.7}$$

$$S = A + B + C \tag{3.8}$$

The physical interpretation of the omega equation (3.6) is that the vertical motion, which is crucial to formation of weather elements such as rain and which is a component of the ageostrophic flow, can be found by knowing properties of the current geostrophic state of the atmosphere. The geostrophic state of the atmosphere is easily obtained from the routine weather charts that are produced by the various meteorological services around the world. It is a crowning glory of quasi-geostrophic theory that it is able to provide a way to deduce the ageostrophic motion, which is extremely difficult to observe, from easily observable variables such as pressure and temperature. Another way to state this property of synoptic scale dynamics is to note that in fact the ageostrophic motion is a consequence of the geostrophic flow and in this sense it should be regarded as intimately linked to that geostrophic or balanced motion. The ageostrophic motion is sometimes said to be unbalanced but this is possibly a misleading term as it is in a certain sense part of the balanced motion.

To find the vertical motion we need to solve the omega equation mathematically which requires a numerical solution method. However it is possible to go a long way towards imagining these solutions by looking at the physics implied by the terms in equation. Consider first the source or forcing, $S = A + B + C$. We refer to A as the vorticity advection term, B as the thermal advection term, and C as the beta term. Each of these source terms is now discussed to give each a dynamical interpretation:

- **Term A**: It is simplest to visualize term A by first defining the vorticity advection, $VA = -\underline{v}_g.\nabla\xi_g$. Note that <u>if</u> the vorticity were to be conserved then $\frac{\partial \xi_g}{\partial t} = VA$; in other words if, for example, VA is positive then, other things being equal, the vorticity will increase at a point. Then we see that $A = -f_0\frac{\partial VA}{\partial z}$.

- **Term B**: Simlarly term B is related to the thermal advection here defined as $TA = -\underline{v}_g.\nabla b'$ so that if for example TA were to be positive then, others things being equal, the temperature increases with time at a given point. Then we see that $B = {\nabla_h}^2 TA$.

- **Term C**: It is easiest to visualize term C by first using the thermal wind equation to obtain: $C = \beta\frac{\partial b'}{\partial x}$.

It is convenient short-hand to write the omega equation as $L[w] = S$ where $L = N^2{\nabla_h}^2 + {f_0}^2\frac{\partial^2}{\partial z^2}$ is a second-order differential operator. Because of the simple second-order derivative form it is often true that $L[w] \propto -w$. In other words if w exhibits a maximum then its second derivative or curvature is negative and if it exhibits a minimum then its curvature is positive.

The constant of proportionality scales as $\frac{N^2}{l^2} + \frac{f^2}{h^2}$ where l and h are typical horizontal and vertical scales of the motion respectively. The Rossby radius of deformation is the horizontal scale, l, which makes both these terms equal ie. $l = \frac{Nh}{f}$. For typical mid-latitude values this means that the typical ratio of horizontal to vertical scales is of order $\frac{N}{f} = 100$.

Thus for many applications we can write $w \propto -S$. So that the sign of that part of the vertical motion due to either term A, B or C is opposite to their signs. In conclusion then we can now predict the vertical motion due to each of these forcing mechanisms:

- **Term A**: If the vorticity advection increases (decreases) with height then there is ascent (descent); i.e. $w \propto \frac{\partial VA}{\partial z}$. Note that there is positive vorticity advection downstream of cyclonic vorticity, such as associated with a trough, or upstream of anticyclonic vorticity.

- **Term B**: This term can be simplified as we note that ${\nabla_h}^2 TA \propto -TA$ using the same reasoning as used for the L operator. Therefore the vertical motion due to term B is such that $w \propto TA$ so that there is ascent (descent) if there is positive (negative) thermal advection.

- **Term C**: the form of term C shows that there is ascent (descent) downstream of warm (cold) air.

For many situations it can be shown that term C is smaller than the other terms and we shall not discuss it further here. It turns out that there can, and often is, a large degree of cancellation between terms A and B. This can be seen using the example of simple cold core disturbance with an upper-level low and a surface high all moving in a uniform zonal flow. It is straight forward to see that the two terms are of opposite sign and because of the uniformity of the flow it is clear that they must cancel exactly. So to consider either in isolation, as is sometimes done in practical applications of the omega equation concept, is misleading. However there is a way to combine these terms to avoid cancellation and this we consider in the next section.

3.2 Omega equation: Q-vector form

Here we seek an accurate yet simple way to avoid the cancellation problem involved when writing the omega equation forcing in advective form. The starting point is the imposition of thermal wind balance as before but we start from the momentum equations rather than the vorticity equation. The geostrophic derivative of the thermal wind shear equation is used:

$$D_g \left(f_0 \frac{\partial v_g}{\partial z} \right) = D_g \left(\frac{\partial b'}{\partial x} \right) \tag{3.9}$$

We will take a constant Coriolis parameter for simplicity in this section. The y-momentum equation is:

$$\frac{Dv}{Dt} + f_0 u + \frac{\partial \phi'}{\partial y} = 0 \tag{3.10}$$

A first approximation is to assume geostrophic balance ie. $f_0 u_g + \frac{\partial \phi'}{\partial y} = 0$ and substituting this into the momentum equation (3.10) gives the second approximation:

$$D_g v_g + f_0(u - u_g) = 0 \tag{3.11}$$

In other words any ageostrophic flow is produced by accelerations of the geostrophic wind. To impose thermal wind as given in equation (3.9) we first take $f_0 \frac{\partial}{\partial z}$ of equation (3.11) to give:

$$D_g \left(f_0 \frac{\partial v_g}{\partial z} \right) = -f_0 \frac{\partial v_g}{\partial z} \frac{\partial v_g}{\partial y} - f_0 \frac{\partial u_g}{\partial z} \frac{\partial v_g}{\partial x} - {f_0}^2 \frac{\partial u_a}{\partial z} \tag{3.12}$$

where $u = u_g + u_a$ and u_a is the x-component of the ageostrophic wind. The right-hand-side of equation (3.12) can be manipulated using the thermal wind shear equations and the non-divergence of the geostrophic wind ie: $\frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0$ to give:

$$D_g \left(f_0 \frac{\partial v_g}{\partial z} \right) = \frac{\partial u_g}{\partial x} \frac{\partial b'}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial b'}{\partial y} - {f_0}^2 \frac{\partial u_a}{\partial z} \tag{3.13}$$

It is convenient to define the first two terms on the right-hand-side of equation (3.13) as:

$$Q_1 = -\left(+ \frac{\partial u_g}{\partial x} \frac{\partial b'}{\partial x} + \frac{\partial v_g}{\partial x} \frac{\partial b'}{\partial y} \right) \tag{3.14}$$

Note that Q_1 is a quantity which depends only on the balanced part of the flow, ie: the geostrophic wind and the temperature. Next we take $\frac{\partial}{\partial x}$ of the thermodynamic equation:

$$D_g \left(\frac{\partial b'}{\partial x} \right) = -\frac{\partial u_g}{\partial x} \frac{\partial b'}{\partial x} - \frac{\partial v_g}{\partial x} \frac{\partial b'}{\partial y} - N^2 \frac{\partial w}{\partial x} \tag{3.15}$$

Now we are in a position to impose thermal wind balance by writing equations (3.13) and (3.15) using the defintion of Q_1:

$$D_g \left(\frac{\partial b'}{\partial x} \right) = Q_1 - N^2 \frac{\partial w}{\partial x} \tag{3.16}$$

$$D_g \left(f_0 \frac{\partial v_g}{\partial z} \right) = -Q_1 - {f_0}^2 \frac{\partial u_a}{\partial z} \tag{3.17}$$

These equations show that the geostrophic motion (via Q_1) tends to change the two parts of the thermal wind balance by equal but opposite amounts. Thus the role of the ageostrophic motion is to maintain the thermal wind balance against this tendency. Equating the left-hand-sides of equations (3.16) and (3.17) we obtain:

$$N^2 \frac{\partial w}{\partial x} - {f_0}^2 \frac{\partial u_a}{\partial z} = 2Q_1 \tag{3.18}$$

Equation (3.18) represents an ageostrophic circulation in the $x - z$ plane forced by the geostrophic flow via Q_1.

An exactly similar derivation applies starting with the x-momentum equation to obtain a circulation equation in the $y - z$ plane:

$$N^2 \frac{\partial w}{\partial y} - {f_0}^2 \frac{\partial v_a}{\partial z} = 2Q_2 \tag{3.19}$$

where the definition of Q_2 is:

$$Q_2 = -\left(\frac{\partial u_g}{\partial y}\frac{\partial b'}{\partial x} + \frac{\partial v_g}{\partial y}\frac{\partial b'}{\partial y} \right) \tag{3.20}$$

The continuity equation for the ageostrophic flow is:

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{3.21}$$

To form the omega equation we now take $\frac{\partial}{\partial x}$ of equation (3.18) added to $\frac{\partial}{\partial y}$ of equation (3.19) using the continuity equation (3.21):

$$N^2 {\nabla_h}^2 w + {f_0}^2 \frac{\partial^2 w}{\partial z^2} = 2\nabla.\underline{Q} \tag{3.22}$$

where $\nabla.\underline{Q} = \frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y}$. Therefore Q_1 is the x-component of the Q-vector and Q_2 is its y-component. We can see that this is the now familiar form of the omega equation but with a compact way of writing the forcing term as the divergence of a vector.

It is possible to simplify the forcing term by using coordinates which lie along (s) and perpendicular (n) to the local tangent to an isentrope or buoyancy contour such that n points towards colder air. Then it can be shown that:

$$\underline{Q} = - \mid \nabla_h b' \mid \underline{k} \times \frac{\partial \underline{v}_g}{\partial s} \tag{3.23}$$

Then translating equation (3.23) into words a prescription for finding the direction of the Q-vector (in the northern hemisphere) is: "moving along an isotherm in the direction of the thermal wind (that is with colder air on your left-hand-side), note the vector change in the geostrophic wind; the direction of the Q-vector is then obtained by rotating this vector through 90° to the right". In addition it is clear from the form of the omega equation that locally large $\underline{Q}$ tend to point towards regions of ascent and away from regions of descent.

Examples of predictions from the Q-vector form of the omega equation which will be given in lectures include: a propagating trough, a jet entrance/exit, a low pressure centre, and a front.

3.3 Summary of Quasi-Geostrophic Theory

The quasi-geostrophic (QG) theory has a simple form for an atmosphere with uniform density using the Boussinesq approximation and height as the vertical coordinate. Note that it is straightforward to extend the theory to include a reference density which varies with height. The theory is basically composed of four equations:

Vortex Stretching:

$$D_g(f + \xi_g) = f_0 \frac{\partial w}{\partial z} \tag{3.24}$$

This equation tells us that the main mechanism for the development of cyclonic systems (ie. low-pressure centres) is the stretching of the Earth's vorticity.

Thermodynamic equation:

$$D_g b' = -wN^2 \tag{3.25}$$

This equation is derived from the conservation of potential temperature (ie. dry adiabatic motion) and shows, for example, that as air rises it cools via adiabatic expansion.

Conservation of QG Potential Vorticity:

$$D_g q = 0 \tag{3.26}$$

This equation shows that in addition to the conservation of potential temperature the air conserves the QG potential vorticity if the motion is frictionless. It follows directly from combining equations (3.24) and (3.25).

Omega Equation:

$$L[w] = 2\nabla.\underline{Q} \tag{3.27}$$

This is a diagnostic equation which, after solution for w, allows the vertical motion to be obtained knowing aspects of the geostrophic flow. Notice that L is a second order differential operator.

These four equations are in fact composed of 2 pairs of equations. The vorticity and thermodynamic equations are one pair. From these equations the potential vorticity and omega equations can be derived. Together they form the second pair of equations. The following are the definitions of the terms appearing in these equations:

QG potential vorticity:

$$q = f + \xi_g + \frac{\partial}{\partial z}\left(\frac{f_0}{N^2} b'\right) \tag{3.28}$$

scaled Laplacian operator:

$$L = N^2 {\nabla_h}^2 + {f_0}^2 \frac{\partial^2}{\partial z^2} \tag{3.29}$$

Q-vector:

$$\underline{Q} = \left(-\frac{\partial u_g}{\partial x}\frac{\partial b'}{\partial x} - \frac{\partial v_g}{\partial x}\frac{\partial b'}{\partial y}, -\frac{\partial u_g}{\partial y}\frac{\partial b'}{\partial x} - \frac{\partial v_g}{\partial y}\frac{\partial b'}{\partial y} \right) \tag{3.30}$$

where N^2 is the square of the Brunt-Vaisala frequency taken to be a function of height only.

3.4 Gravity waves and adjustment

In this discussion of quasi-geostrophic theory there is an implicit assumption that there is a mechanism in the atmosphere to restore geostrophic balance if there are small departures from that balance. In other words how does the atmosphere make the continual small adjustments necessary to the mass field so as to keep the pressure field near to balance with the wind field? The answer is that there is a mechanism and it involves a basic mode of oscillation of the atmosphere with the production of inertia-gravity waves (IGW). All atmospheric flows which are subject to accelerations, that is those which are not in precise geostrophic or hydrostatic balance, exhibit IGW to a lesser or greater extent. Examples of situations in which IGW are generated include: flow over orography, convection, stratified boundary layer motion, and at jet entrances and exits. The typical horizontal and vertical scales of the IGW that are so generated will often be related to the forcing mechanism, for example the width of a mountain range. However the forcing will produce a spectrum of waves with a variety of wavelengths perhaps centred on that forcing scale. Here a Fourier decomposition of the wave spectrum will be made and the dispersion relation determined by relating the wave frequency to the wavelengths in that spectrum. The full wave response can then, in principle, be determined by a Fourier combination of the modes. This step is in practice mathematically involved and will not be considered here.

It is simplest to examine the properties of the adjustment by deriving the properties of gravity waves as a small departure from a geostrophically balanced flow $\overline{U}$. Also to simplify the derivation we take waves which are two-dimensional such that $\frac{\partial}{\partial y} = 0$. The waves are assumed to be of infinitesimal amplitude, denoted by a $'$ hereafter, and exist in a flow with a constant wind $\overline{U}$ and static stability N^2. The Boussinesq equations can then be written after linearization (ie. neglecting advective terms that involve products of wave quantities) in the following way:

$$\overline{D}u' - fv' + \frac{\partial \phi'}{\partial x} = 0 \tag{3.31}$$

$$\overline{D}v' + fu' = 0 \tag{3.32}$$

$$n\overline{D}w' + \frac{\partial \phi'}{\partial z} = b' \tag{3.33}$$

$$\overline{D}b' + w'N^2 = 0 \tag{3.34}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{3.35}$$

where $\overline{D} = \frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}$, $\phi' = \frac{p'}{\rho_0}$ is the perturbation geopotential, and n is a tracer parameter which is zero if we make the hydrostatic approximation but unity otherwise. Notice that the geostrophic and hydrostatic parts due to the mean flow cancel so do not appear in these wave equations.

A separable solution is sought for each variable of the form: $\chi' = \hat{\chi}(z)e^{i(kx-\nu t)}$. This is a single Fourier component and has the property that $\frac{\partial}{\partial t} = -i\nu$ and $\frac{\partial}{\partial x} = ik$. It is understood that the real part of any variable represents the physically observable flow. Substituting this solution into the linear equations gives:

$$\phi' = \left(\frac{\nu}{k} - \overline{U}\right)u' - i\frac{f}{k}v' \tag{3.36}$$

$$v' = -i\frac{f}{(\nu - k\overline{U})}u' \tag{3.37}$$

$$w' = \frac{i}{n(\nu - k\overline{U})}\left(b' - \frac{\partial \phi'}{\partial z}\right) \tag{3.38}$$

$$b' = -\frac{i}{(\nu - k\overline{U})}N^2 w' \tag{3.39}$$

$$u' = \frac{i}{k}\frac{\partial w'}{\partial z} \tag{3.40}$$

It is usual to obtain a single equation in one of the wave variables by elimination and here we take that to be the vertical velocity. Substituting into equation (3.36) for u' from equation (3.40) and for v' from equation (3.37), and then substituting into equation (3.38) for b' from equation (3.39) and for ϕ' from equation (3.36) gives the wave equation:

$$\frac{\partial^2 w'}{\partial z^2} + k^2\left\{\frac{N^2 - (\nu - k\overline{U})^2 n}{(\nu - k\overline{U})^2 - f^2}\right\}w' = 0 \tag{3.41}$$

This is the IGW equation and it can be solved given suitable boundary conditions. Imagine a layer unbounded in the vertical so that equation (3.41) has a solution with $w' \propto e^{i(kx+mz-\nu t)}$, where the vertical wavenumber m satisfies the equation:

$$m^2 = k^2\left\{\frac{N^2 - (\nu - k\overline{U})^2 n}{(\nu - k\overline{U})^2 - f^2}\right\} \tag{3.42}$$

As suggested earlier it is the aim to obtain the wave frequency ν as a function of the wavenumbers and this can be found by rearrangement of equation (3.42):

$$\nu = k\overline{U} \pm \sqrt{\frac{f^2 m^2 + N^2 k^2}{m^2 + nk^2}} \tag{3.43}$$

This is the dispersion equation for inertia-gravity waves. The waves are clearly dispersive as $c = \frac{\nu}{k}$, the trace speed, is a function of k. The frequency also depends on the Coriolis parameter and the static stability. A simplification is to define an aspect ratio $\alpha = \frac{k}{m} = \frac{L_z}{L_x}$ which is the ratio of the vertical to horizontal wavelengths of the IGW. The dispersion relation then becomes:

$$\nu = k\overline{U} \pm \sqrt{\frac{f^2 + \alpha^2 N^2}{1 + \alpha^2 n}} \tag{3.44}$$

It is illuminating to consider the so-called Doppler-shifted frequency, $\nu_D = \nu - k\overline{U}$, which is the frequency relative to the propagating wave. Taking the fully non-hydrostatic case with $n = 1$ the two limiting cases are:

$$\alpha \rightarrow 0 \Rightarrow L_z \ll L_x, \nu_D \rightarrow \pm f \tag{3.45}$$

$$\alpha \rightarrow \infty \Rightarrow L_z \gg L_x, \nu_D \rightarrow \pm N \tag{3.46}$$

therefore the wave frequency is bounded by: $| f | \leq | \nu_D | \leq N$. These two limits are referred to as the inertia and gravity wave limits.

In mid-latitudes this suggests a range of wave periods ($= \frac{2\pi}{|\nu_D|}$) of 17h to 10 min which shows the wide range of IGW frequencies.

In large-scale numerical models of the atmosphere such as the current generation of weather prediction models the hydrostatic approximation is made. We can assess the impact of this assumption on the IGW dispersion relation by setting $n = 0$ in the equations derived thus far. The limits on the frequency then become:

$$\alpha \rightarrow 0 \Rightarrow L_z \ll L_x, \nu_D \rightarrow \pm f \tag{3.47}$$

$$\alpha \rightarrow \infty \Rightarrow L_z \gg L_x, \nu_D \rightarrow \pm \infty \tag{3.48}$$

Thus the hydrostatic approximation distorts the short horizontal scale waves significantly. It is clear that this approximation is only valid for $\alpha \ll 1$ or $L_x \gg L_z$. Note that these wavelengths apply to the motion and are not given by the numerical model grid-lengths.

An important property of any wave is the associated group-velocity vector which determines the direction and rate of wave energy propagation. In the case of no mean flow the group velocity is given by:

$$\underline{c}_g = \left(\frac{\partial \nu}{\partial k}, \frac{\partial \nu}{\partial m} \right) = \frac{d\nu}{d\alpha} \left(\frac{\partial \alpha}{\partial k}, \frac{\partial \alpha}{\partial m} \right) \tag{3.49}$$

From the definition of α and ν this gives:

$$\underline{c}_g = \frac{N^2 - f^2}{m^2(1+\alpha)^2} \left\{ \frac{k}{\nu}, -\alpha^2 \frac{m}{\nu} \right\} \tag{3.50}$$

It can be seen that the horizontal component of the group velocity has the same sign as the horizontal phase propagation but its vertical component has the opposite sign to the vertical phase propagation. Another implication of equation (3.50) is that the group velcity vector lies parallel to the wave phase lines. This somewhat bizarre property that energy propagates at right angles to the direction of the phase propagation has important implications, for example, for flow over mountains and will be discussed later in the course.

Chapter 4

Waves and Instabilities

4.1 Edge waves

4.1.1 Barotropic flow

In a two-dimensional barotropic flow we can discuss a basic building block of atmospheric dynamics called an edge wave by considering a two-dimensional interface between low and high vorticity. For example consider a zonally oriented interface between air of uniform westerly flow (ie. zero relative vorticity) to the north of the interface and air with uniform cyclonic relative vorticity to the south of the interface. If perturbed this interface will support a propagating edge wave. Imagine a northward bulge in the interface. This represents a positive vorticity anomaly in the air to the north of the interface. Using action-at-a-distance it will have a cyclonic flow with southerly flow ahead and northerly flow behind the anomaly. These flows will distort the interface both ahead of and behind the original bulge. The distortion is such that the bulge propagates from west to east along the interface: see Figure (4.1). This is an edge wave and it is our intention to show that complex systems such as cyclones are composed of such waves.

It can be shown that an expression for the speed of this barotropic edge wave is given by:

$$c = \overline{U}_b - \frac{\Delta\zeta}{2k} \tag{4.1}$$

where $\overline{U}_b$ is the mean flow at the interface, $\Delta\zeta$ is the jump in vorticity at the interface and k is the horizontal wavenumber of the edge wave. So the wave moves more slowly than the mean flow at that location by an amount that depends on the strength of the mean vorticity transition.

4.1.2 Baroclinic flow

We can extend the concept of an edge wave to a baroclinic three-dimensional flow. In Chapter 2 the quasi-geostrophic potential vorticity was discussed and the temperature (and wind) attributable to a localised anomaly was found. In figure (2.1) the isentropes, in a vertical

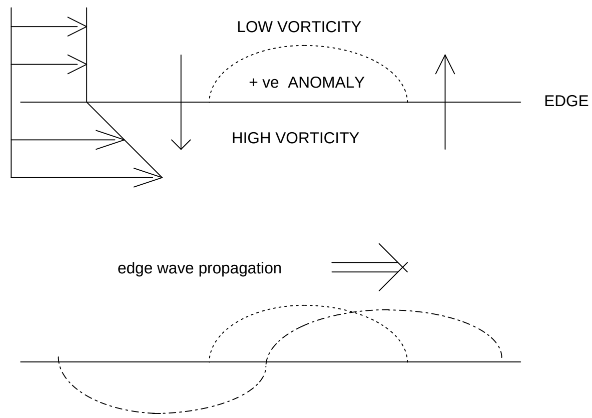

Figure 4.1: *A schematic of wave propagation along an edge between high and low vorticity. An initial bulge in the interface induces a meridional flow thus distorting the interface further leading to wave propagation*

cross-section, associated with a q' ball anomaly were shown. This diagram can be used in a variety of ways. If we imagine a horizontal line drawn to represent the ground through the centre of the potential vorticity anomaly in figure (2.1) and consider the atmosphere in the plane above this line, we obtain a structure with a hemisphere of positive potential vorticity with no temperature signature at the ground but with cyclonic circulation there. Above the ground there is a warm thermal anomaly. Equally if we imagine a negative potential vorticity anomaly and consider the plane beneath this horizontal line then there will also be warm air but now with anticyclonic circulation beneath. Therefore we can add a hemisphere of positive q' at the ground and a hemisphere of negative q' at the tropopause to obtain a single mid-tropospheric warm core disturbance in which cyclonic circulation at the ground decreases with height to become anticyclonic aloft.

The previous example has no temperature anomalies at the two bounding surfaces. However this diagram can also be used to show that if there is a temperature anomaly at the Earth's surface the balanced flow associated with it must be cyclonic. Imagine in figure (2.1) a horizontal line representing the ground drawn just above the top of the q' ball anomaly. If we examine the half-plane bounded below by this line then there is a local zone of warm air and cyclonic circulation even though in this half-plane there is no potential vorticity anomaly. The associated potential vorticity anomaly can be thought of as existing "underground"! So it is clear that surface temperature anomalies are associated with geostrophic circulation in the following way: *warm (cold) air at the ground is associated with cyclonic (anticyclonic) circulation*. Similarly placing the horizontal line just below the potential vorticity anomaly in figure (2.1) tells us how to associate temperature anomalies and wind anomalies on the tropopause: *warm (cold) air at the tropopause is associated with anticyclonic (cyclonic) circulation.* Notice the opposite correlation between temperature and circulation at the two boundaries: see Figure (4.2).

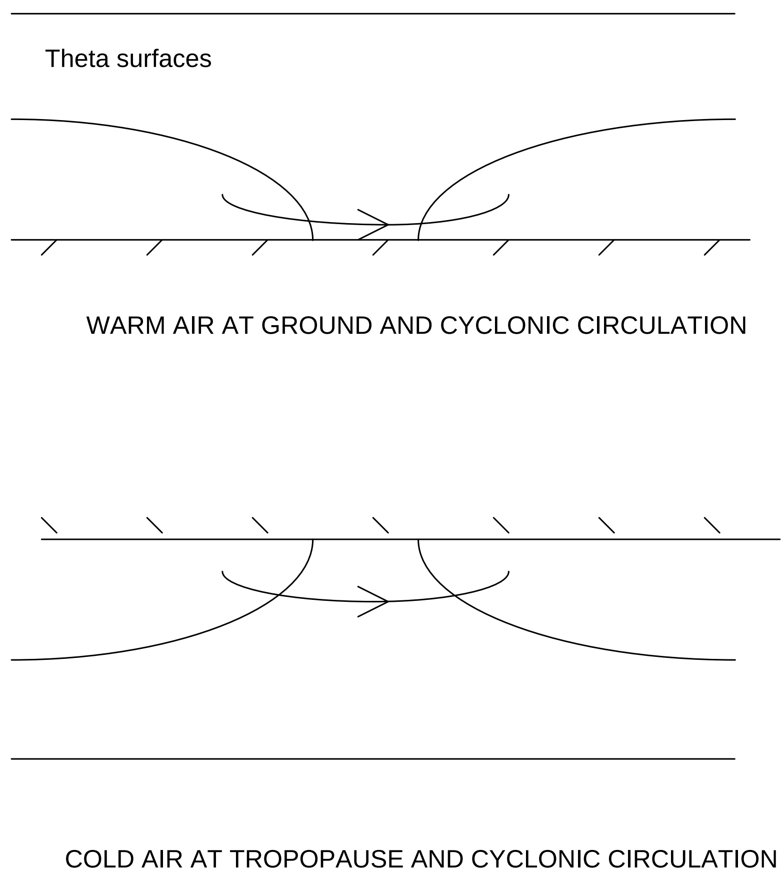

Figure 4.2: *Association between temperature anomalies and circulation at the ground and the tropopause*

In the presence of a synoptic scale horizontal temperature gradient such surface temperature anomalies will propagate in the form of what are known as edge waves, as we will now demonstrate. We first impose the condition that the atmosphere contains no perturbation potential vorticity anomalies ie:

$$q' = {\nabla_h}^2\psi + \frac{\partial}{\partial z}\left(\frac{{f_0}^2}{N^2}\frac{\partial \psi}{\partial z}\right) = 0 \tag{4.2}$$

Solutions of equation (4.2) exist of the following form:

$$\psi = \hat{\psi}(t) sin(ly) sin(kx) e^{\pm\frac{\kappa N z}{f_0}} \tag{4.3}$$

where the boundary, or edge, in question is located at $z = 0$ and $\kappa = \sqrt{k^2 + l^2}$ is the total horizontal wavenumber. Note that the choice of sign depends on which boundary the wave resides, ie. for a wave at the ground then the minus sign is used whereas for a wave with maximum amplitude at the tropopause then the positive sign is used. Thus in either case the wave decays away from the boundary. We now use the thermodynamic equation to describe the wave propagation at the boundary where the vertical velocity is zero, ie:

$$\frac{\partial b'}{\partial t} + \overline{U}_b \frac{\partial b'}{\partial x} + v' \frac{\partial \bar{b}}{\partial y} = 0 \tag{4.4}$$

here $\overline{U}_b$ is the mean flow at the boundary and all prime quantities refer to the small amplitude wave. Note that we assume a uniform meridional basic state temperature gradient, $\frac{\partial \bar{b}}{\partial y}$, in thermal wind balance with the mean flow $\overline{U}$. Using geostrophic and hydrostatic equations this can be written as:

$$\frac{\partial^2 \psi}{\partial t \partial z} + \overline{U}_b \frac{\partial^2 \psi}{\partial x \partial z} + \frac{1}{f_0}\frac{\partial \bar{b}}{\partial y}\frac{\partial \psi}{\partial x} = 0 \tag{4.5}$$

A time-dependent solution of equation (4.5) which is consistent also with equation (4.3) is given by:

$$\psi = \psi_0 sin(ly) sin(k(x - ct)) e^{\pm\frac{\kappa N z}{f_0}} \tag{4.6}$$

where ψ_0 is a constant and equation (4.6) is a solution of equations (4.5) and (4.3) if the phase speed c satisfies:

$$c = \overline{U}_b \pm \frac{\frac{\partial \bar{b}}{\partial y}}{\kappa N} \tag{4.7}$$

This shows that the thermal edge wave at the boundary propagates either slower or faster than the mean flow at that boundary. Notice that typically there is colder air to the north so that $\frac{\partial \bar{b}}{\partial y} < 0$ so that at the ground (where the minus sign in equation (4.7) applies) the wave moves faster than the mean flow whereas on the tropopause (where the positive sign in equation (4.7) applies) the wave moves slower than the mean flow. Also the phase speed is proportional to the wavelength. (Note that the solution in equation (4.6) substituted into the thermodynamic equation, valid away from the boundaries, implies that there will be some vertical motion associated with the wave.)

Using thermal wind balance and defining a Rossby depth scale $H_R = \frac{f_0}{\kappa N}$ equation (4.7) can be written as:

$$c = \overline{U}_b \mp H_R \frac{d\overline{U}}{dz} \tag{4.8}$$

This shows that the wave moves at the speed of the mean flow at a vertical distance H_R from the boundary. This height is called the steering level of the wave. Typical values might be: $\kappa = \frac{2\pi}{1000km}$, $N = 10^{-2}s^{-1}$, and $f_0 = 10^{-4}s^{-1}$ giving $H_R = 1.6km$.

4.2 Horizontal coupling - barotropic instability

A key implication of the existence of the edge waves is that an instability can develop. The simplest example of this is barotropic instability in which we imagine a zonal strip of non-zero relative vorticity of meridional extent L. Then the two edges of the vorticity strip can support an edge wave and because of horizontal action-at-a-distance the two edge waves can interact with one another. This interaction leads to the possibility of barotropic instability.

The model described in this section is a simple example illustrating barotropic instability. It was first studied by Rayleigh (1880) and solved in a quasi-geostrophic context in Gill (1982). We use an f-plane (O,x,y). The basic state wind is zonal and depends only on y. It is defined in three regions :

$$\begin{cases} -\frac{L}{2} < y < \frac{L}{2} & \overline{U} = \Lambda y \\ y > \frac{L}{2} & \overline{U} = \Lambda \frac{L}{2} \\ y < -\frac{L}{2} & \overline{U} = -\Lambda \frac{L}{2} \end{cases}$$

$\Lambda = \frac{\partial \overline{U}}{\partial y}$ is the uniform horizontal shear vorticity of the basic state in the central strip region.

The equations to specify the instability are derived from the conservation of vorticity: $\frac{D\zeta}{Dt} = 0$. The flow is divided up into a basic state denoted by an overbar and a perturbation denoted by a prime ie: $\zeta = \overline{\zeta} + \xi'$ where $\overline{\zeta} = f - \frac{\partial \overline{U}}{\partial y}$, ξ' is the perturbation relative vorticity, $u = \overline{U} + u'$, and $v = v'$. Linearization of the vorticity equation then gives:

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)\xi' + v'\frac{\partial \overline{\zeta}}{\partial y} = 0 \tag{4.9}$$

As the flow is two-dimensional then we can use a streamfunction for the winds which we can assume is wavelike in the x-direction and has an exponential time variation, ie: $\psi \propto e^{(ikx+\sigma t)}$, such that σ is the growth rate of the instability. Then the linearized vorticity equation can be written as:

$$\frac{\partial^2 \psi}{\partial y^2} - \psi\left(\frac{\partial^2 \overline{U}}{\partial y^2}\frac{k}{(k\overline{U} - i\sigma)} + k^2\right) = 0 \tag{4.10}$$

To solve this equation matching conditions at the edges of the vorticity strip have to be found. These are continuity of streamfunction, ψ, and the second condition can be obtained by integrating equation (4.10) across the two strip edges. To find the second boundary condition we integrate across an edge of the vorticity strip located at $Y\ (= \pm\frac{L}{2})$, from $Y - \epsilon$ to $Y + \epsilon$, where ϵ is a very small distance. Noting that ψ and $\overline{U}$ are continuous at the interface we obtain upon integration:

$$\frac{\partial \psi}{\partial y}|_{Y+\epsilon} - \frac{\partial \psi}{\partial y}|_{Y-\epsilon} = \frac{k\psi \left(\frac{\partial \overline{U}}{\partial y}|_{Y+\epsilon} - \frac{\partial \overline{U}}{\partial y}|_{Y-\epsilon}\right)}{(k\overline{U} - i\sigma)} \tag{4.11}$$

This condition must be applied at both edges of the vorticity strip along with continuity of ψ. A solution which satisfies the basic equation (4.10) and the boundary conditions is :

$$\begin{cases} \psi(x,y,t) = (a\ e^{ky} + b\ e^{-ky})e^{(ikx+\sigma t)} & |y| < \frac{L}{2} \\ \psi(x,y,t) = A\ e^{-ky}\ e^{(ikx+\sigma t)} & y > \frac{L}{2} \\ \psi(x,y,t) = B\ e^{ky}\ e^{(ikx+\sigma t)} & y < -\frac{L}{2} \end{cases}$$

Substituting this solution into the four boundary conditions allows us to determine three of the constants a, b, A and B in terms of the other one and allows the growth rate equation to be determined:

$$\sigma \quad = \quad \frac{|\ \Lambda\ |}{2}\sqrt{e^{-2kL} - (1-kL)^2} \tag{4.12}$$

The maximum value for the growth rate is $\sigma_{max} = 0.2\frac{d\overline{U}}{dy}$ when $kL = 0.8$. This means that the wavelength of this maximum growing disturbance is about $8L$. Notice that a necessary condition for the instability is that there are 2 edge waves in close enough proximity to allow interaction. This aspect is apparent from equation (4.12) as for waves short relative to the strip width L (ie: large kL) there is no instability.

A way to state the instability criterion is that there must be a local maximum or minimum of vorticity. Then the 2 edge waves can have an equal phase speed so keeping in step with one another. At $y = \frac{L}{2}$ we expect $c < \overline{U}(\frac{L}{2})$ whereas at $y = -\frac{L}{2}$ we expect $c > \overline{U}(-\frac{L}{2})$. In fact due to the symmetry of the basic state flow the instability has a zero phase speed which is consistent as $\overline{U}(-\frac{L}{2}) < 0 < \overline{U}(\frac{L}{2})$.

4.3 Vertical coupling - baroclinic instability

Vertical coupling in a baroclinic flow between edge waves at the ground and at the tropopause leads to baroclinic instability. To provide a non-mathematical description of this vertical interaction we assume that there is a mean zonal flow in thermal wind balance with the north-south temperature gradient. It is simplest to look at the dynamics in the frame of reference moving with the instability which turns out to be the mid-tropospheric flow. Visualization of the dynamics will be considered as a 4 step process. These steps are not intended as a time sequence but are simply a way to build-up the picture from the building blocks outlined in previous sections. The essential dynamical tools for this particular description are the vorticity, thermodynamic and omega equations. (The alternative description using the potential vorticity perspective is, of course, valid and entirely complementary. It is explored in the first question on Problem Sheet 3.)

1. *Vertical motion due to upper-level trough*: Consider an upper-level trough which can be thought of as an anomaly of positive vorticity at tropopause level. The trough is advecting relative to the lower troposphere due to the sheared zonal flow. Using the omega equation we know that in the upper troposphere there will be ascent ahead and descent behind the trough. (See equations (3.6) and (3.7) and retaining the vorticity advection forcing.)

2. *Generation of low-level vorticity*: The ascent causes stretching of vortex tubes in the lower troposphere ahead of the upper trough and from the vorticity equation, equation (3.24), this leads to vorticity generation at the surface underneath the ascent zone. Equally there will be generation of anticyclonic vorticity beneath the descent behind the trough. So a surface low-high dipole forms. Note that the low pressure tilts to the west with height.

3. *Low-level vertical motion*: The low-level vorticity pattern is advecting relative to the upper-level pattern and so from the omega equation we expect that in the lower troposphere there will be ascent to the east of the low and descent between the low and high. The location of this vertical motion pattern results from the low-level (relative) zonal flow being from the east in this relative reference frame. The upper and lower tropospheric patterns link to form zones of ascent and descent which tilt to the west with height. Notice that the descent at low-levels between the surface low and high is directly beneath the upper level low. Hence vortex stretching in the upper troposphere in this location will amplify the original upper-level trough. This amplification is the reason why such a flow is baroclinically unstable.

4. *Thermal advection contribution*: The vorticity pattern at the surface results, via action-at-a-distance, in southerly flow ahead of the low and northerly flow between the low and high (see equation (2.30)). This advects warm and cold air into the plane in question due to the north-south temperature gradient (see the thermodynamic equation (4.4)). Also these surface thermal anomalies are advected towards the west by the zonal flow. The net result of this thermal advection is that the warm air lies just to the east of the surface low. Similar advection takes place at the tropopause but now the anomalies are advected towards the east. It can be seen that as a consequence of advection on the two boundaries the temperature pattern tilts to the east with height.

It is easy to see from the final structure of this baroclinic wave that in the middle troposphere there is warm air rising and moving polewards with cold air moving downwards and southwards. Hence baroclinic waves transport heat northwards thus acting to reduce the north-south temperature gradient being built up by radiative forcing.

4.4 Eady model: type A cyclogenesis

The Eady model is a mathematical description of the baroclinic instability we have just qualitatively described in the previous section. It is used as one of the explanations for cyclogenesis in the extra-tropics. The model consists of a basic state and a small amplitude perturbation which will represent the baroclinic wave or cyclone. The basic state is a zonal flow with a constant vertical shear but with no horizontal shear. From thermal wind balance we see that this implies a constant meridional temperature gradient. The basic state also assumes a constant static stability and Coriolis parameter. We will also take uniform density. Hence $\overline{U} = Az$ where A is the constant shear and $f_0 A$ is the constant north-south buoyancy gradient of the basic state. The basic state therefore has a constant QG potential vorticity equal to f_0.

To simplify the analysis we assume that the perturbation is independent of the north-south coordinate ie: $\frac{\partial}{\partial y} = 0$. The model can be solved relaxing this constraint but the essential physics is unchanged. As we are using the quasi-geostrophic equations the horizontal flow is assumed to be in geostrophic balance. A starting point for the Eady model is the QG potential vorticity equation: $D_g q = 0$. As the mean flow potential vorticity is a constant then this reduces to $D_g q' = 0$. A linearization is now performed wherein $u = \overline{U} + u'_g$ and $v = v'_g$. Hence the linearized potential vorticity equation becomes:

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right) q' = 0 \tag{4.13}$$

There are a special class of solutions of this equation, $q' = 0$, which can be contrasted with those which we will discuss in the next section which have $q' \neq 0$ but also satisfy equation (4.13). Here we take $q' = 0$ which implies, given that the perturbations are geostrophic:

$$\nabla_h{}^2\psi + \frac{\partial}{\partial z}\left(\frac{f_0{}^2}{N^2}\frac{\partial \psi}{\partial z}\right) = 0 \tag{4.14}$$

This second-order differential equation requires two boundary conditions to allow it to be solved. These conditions arise from assuming that the motion takes place in a layer of depth H bounded below by the ground and above by the tropopause where it is assumed the vertical velocity is zero. The QG thermodynamic equation applied at these two boundaries and linearised then provide the required boundary conditions (remember that there is a basic state meridional temperature gradient):

$$\left(\frac{\partial}{\partial t} + \overline{U}\frac{\partial}{\partial x}\right)\frac{\partial \psi}{\partial z} = A\frac{\partial \psi}{\partial x} \tag{4.15}$$

applied at $z = 0$ and $z = H$. This equation states that at the boundaries there is zonal temperature advection by the basic flow and meridional temperature advection by the perturbation. This featured in step 4 of the qualitative model developed in the last section.

It is convenient to introduce temporary non-dimensional variables to ease the mathematical complexity of the derivation:

$$\hat{z} = \frac{z - \frac{H}{2}}{H} \quad (4.16)$$

$$\hat{x} = \frac{x - \frac{AH}{2}t}{L_R} \quad (4.17)$$

$$\hat{t} = \frac{fA}{N}t \quad (4.18)$$

where $L_R = \frac{NH}{f}$ is the Rossby radius of deformation and a caret indicates a non-dimensional quantity. The basic equations then become:

$$\frac{\partial^2 \psi}{\partial \hat{x}^2} + \frac{\partial^2 \psi}{\partial \hat{z}^2} = 0 \quad (4.19)$$

$$\left(\frac{\partial}{\partial \hat{t}} + \hat{z}\frac{\partial}{\partial \hat{x}}\right)\frac{\partial \psi}{\partial \hat{z}} - \frac{\partial \psi}{\partial \hat{x}} = 0 \quad at \quad \hat{z} = \pm\frac{1}{2} \quad (4.20)$$

Growing solutions of equation (4.19), which grow in time, are of the form:

$$\psi = \psi_0 \left(cosh(\hat{k}\hat{z})cos(\hat{k}\hat{x}) - \hat{r}.sinh(\hat{k}\hat{z})sin(\hat{k}\hat{x})\right) e^{\hat{\sigma}\hat{t}} \quad (4.21)$$

where the non-dimensional wavenumber is $\hat{k} = kL_R$, the non-dimensional growth rate is $\hat{\sigma} = \frac{N}{fA}\sigma$ and ψ_0 is an arbitrary amplitude. Substituting this solution into equation (4.20) gives, after some algebra, the growth rate equation:

$$\hat{\sigma} = \sqrt{\left(\frac{\hat{k}}{2} - tanh\frac{\hat{k}}{2}\right)\left(coth\frac{\hat{k}}{2} - \frac{\hat{k}}{2}\right)} \quad (4.22)$$

Making the substitution also allows us to obtain an expression for the ratio of the edge wave amplitudes, $\hat{r}$:

$$\hat{r} = \sqrt{\frac{1 - \frac{\hat{k}}{2}tanh\frac{\hat{k}}{2}}{\frac{\hat{k}}{2}coth\frac{\hat{k}}{2} - 1}} \quad (4.23)$$

Converting back to dimensional parameters, in equation (4.22), we obtain the Eady model growth rate equation:

$$\sigma = \frac{fA}{N}\sqrt{\left(\frac{kL_R}{2} - tanh\frac{kL_R}{2}\right)\left(coth\frac{kL_R}{2} - \frac{kL_R}{2}\right)} \quad (4.24)$$

If the growth rate is real then the wave amplitude has a constant shape but it increases exponentially in time. This is referred to as growth. Whereas if σ is imaginary then the wave amplitude simply oscillates in time and there is no increase from the infinitesimal amplitude we have assumed. (Note that in this case the solution given in equation (4.21) needs to be modified.) This is referred to as a neutral wave. The maximum growth rate occurs for

$\hat{\sigma} = 0.31$ at a wavenumber of $\hat{k} = 1.6$. The short waves ($kL_R > 2.4$) are neutral whereas the longer waves ($kL_R < 2.4$) are "baroclinically" unstable. Due to the choice of the non-dimensional height variable we see that the growing solutions have zero phase speed. This means that the wave moves with the basic state wind at the mid-level of the layer. Neutral waves move at the speed of the basic state flow at a level somewhere between the mid-level and the boundary depending on the wavelength.

The Eady wave structure confirms the deductions we made on vertically interacting baroclinic edge waves. Namely the pressure field tilts 90°, or $\frac{\lambda}{4}$, westwards between the ground and the tropopause, the vertical velocity tilts approximately 45° westwards with height and the temperature wave tilts eastwards with height. The maximum of warm air at the boundary lies 21° to the east of the low centre. At mid-levels the correlations are perfect to transport warm air northwards and upwards and cold air southwards and downwards.

The Eady wave is the simplest and most elegant example of a baroclinic wave and it is used as a basic explanation for the growth mechanism of cyclones in the extra-tropics. The solutions are referred to as being "normal modes" and they have the property of retaining a constant shape but simply growing in amplitude in time. So from arbitrary initial perturbations the fastest growing Eady mode will emerge as the dominant structure. The rate at which this normal mode establishes itself from the arbitrary initial conditions is **not** the same as the growth rate and in numerical calculations it found to be many times the timescale implied by the growth rate. This method of development is called type A to distinguish it from the mechanism we will discuss in the next section. It does not need any special pre-existing structure to force cyclogenesis; rather it relies on the growth of "noise".

4.5 Non-modal growth: type B cyclogenesis

It is clear from synoptic charts that there are frequently upper-level short wave trough features moving from the continent of the USA over the western side of the Atlantic storm-track. We know from the QG vorticity equation that such propagating troughs will lead to ascent and low-level cyclonic development. This is referred to as type B cyclogenesis as it needs a pre-existing structure to produce the growth of the cyclone. How can we describe this process within the context of the QG system and concepts? We can start by re-examining equation (4.13) which encompasses the conservation of QG potential vorticity. The Eady model takes solutions which have zero QG potential vorticity. However if there is a pre-existing structure it will have a non-zero q'. An upper-level trough, for example, is associated with an undulation in the tropopause which along an isentropic surface appears as a positive q' anomaly.

Solutions of equation (4.13) exist of the type $q' = q'(x - \overline{U}t)$. An alternative way to write this functional dependence is to note that if $\overline{U} = Az$ then it can be written as: $q' = q'(x - \alpha(t)z)$ where $\alpha = \alpha_0 + At$. Notice that this solution has a vertical tilt, ie: $\frac{dz}{dx} = \alpha^{-1}$ and the "initial" vertical tilt at $t = 0$ is given by ${\alpha_0}^{-1}$. As an example suppose that $q' = \hat{q}\sin(k(x - \alpha z))$ where $\hat{q}$ is the amplitude of the potential vorticity anomaly. Equation (2.20), defining the QG potential vorticity in terms of the geostrophic streamfunction, then has the following solution:

$$\psi = -\frac{\hat{q}}{k^2\left(1+\frac{f^2}{N^2}(\alpha_0+At)^2\right)}\sin\left(k(x-(\alpha_0+At)z)\right) \tag{4.25}$$

Several aspects of these solutions are worthy of note. If $\alpha_0 < 0$ then the potential vorticity and streamfunction at the initial time tilt to the west with height, ie: against the basic state wind shear. As time progresses the q' and ψ field become progressively more vertical and achieve the vertical at a time $-\frac{\alpha_0}{A}$. They then tilt down shear or to the east with height. Contrast this behaviour with the constant shape Eady wave. In addition the amplitude of the streamfunction (but not the potential vorticity) increases before ultimately decaying in the long time limit to zero. Hence this solution does represent a growing pattern, at least for a certain length of time. It is clear that for certain values of the determining parameters then growth via this type B mechanism is greater than that of the Eady normal mode. However this growth is limited to a finite time interval. Another name for this mechanism is "non-normal" mode, or non-modal, growth.

Chapter 5

Mesoscale Weather Systems

5.1 Frontogenesis and semi-geostrophy

5.1.1 Fronts

There are many different types of front in the atmosphere. The word front is a general one relating to an imaginary interface between two air-masses of markedly different properties. The type of front we will be focussing on here is created within an evolving extra-tropical cyclone. Such fronts are referred to as either cold, warm, or occluded fronts. The adjective cold or warm relates to whether cold air replaces warm or vice versa. An occluded front or occlusion occurs where the main cold and warm fronts are located above the surface. Usually cold and warm fronts delineate a warm sector. The cold, warm and occluded fronts meet at the so-called triple point.

Another way to classify fronts depends on whether there is ascent or descent along the frontal surface. These are referred to as ana and kata fronts respectively. Usually ana fronts are more active in weather terms than kata fronts. Also we can distinguish surface fronts and upper fronts. Upper fronts are large temperature gradients occuring in the upper troposphere and at the tropopause. Sometimes on satellite pictures the cloud edges indicate the position of fronts and in a split front case the upper front lies on the warm air side of the cold front. The more usual situation is when the upper front is on the cold air side of the surface front.

Fronts were so named by the Bergen School of meteorologists who studied with Vilhelm Bjerknes in Bergen, Norway in the 1920's. Amongst the key figures were Jack Bjerknes (Vilhelm's son), Tor Bergeron and Halvor Solberg. They used surface data to catalogue (and forecast) the passage of these air mass transitions and used the word front as an analogy with battle fronts in the First World War. They imagined atmospheric fronts as separating warring air-masses! However even in the 19th century it was realised that if there was a so-called discontinuity in the atmosphere between fluids of different densities then the interface must slope with height. The great German scientist Hermann von Helmholtz discusses this in a famous paper in 1888 and this was more fully applied to the atmosphere by Max Margules, an Austrian meteorologist, in 1904. The formula for frontal slope derived by Margules is still in use today. In 1920 Bergeron discusses the mechanisms for frontal formation including the so-called Bergeron deformation flow.

In two papers published independently by Eliassen and Sawyer in 1956 the explanation for vertical motion at fronts was elucidated and an equation for the cross-frontal circulation was derived. A mathematical theory of frontogenesis was published by Hoskins around 1980 and a more accurate version of quasi-geostrophic theory, known as semi-geostrophic theory, was derived. This provides the framework for the discussion of fronts in this course.

5.1.2 Frontal geostrophy approximation

Fronts are extremely anisotropic phenomena in the horizontal plane. The cross-front extent might be of order $100km$ whilst the along-front dimension is of order $1000km$. This anisotropy has important implications for the dynamics of fronts. For example the Rossby number can be estimated using the cross-front distance of 100km and a typical wind speed of $20ms^{-1}$ which gives a Rossby number of 2. Therefore we cannot neglect accelerations as we have done implicitly in quasi-geostrophic theory which applies only when $R_o \ll 1$.

The frontal geostrophy approximation is a way to improve upon the QG system. Suppose the front lies north-south and has a cross-front horizontal scale of l and an along-front scale of L. From observations it is clear that $l \ll L$ and $u < v$. Therefore the substantial derivative scales as: $\frac{D}{Dt} \sim \frac{u}{l}$. Consider first the along-front or y momentum equation. The appropriate scaling of terms for estimation of the Rossby number is given by:

$$\frac{\frac{Dv}{Dt}}{fu} \sim \frac{\frac{u}{l}v}{fu} \sim \frac{v}{fl} \sim 2 \tag{5.1}$$

Thus the acceleration term cannot be neglected in this along-front equation. Performing a similar scale analysis for the x-momentum equation:

$$\frac{\frac{Du}{Dt}}{fv} \sim \frac{\frac{u}{l}u}{fv} \sim \frac{v}{fl}\frac{u^2}{v^2} \ll 1 \tag{5.2}$$

The inequality results from the fact that $u < v$ and often $u \ll v$. Hence it is a good approximation to neglect accelerations in the cross-front momentum equation so that $v = v_g$. Hence we arrive at the frontal geostrophy approximation which is that the along-front wind is close to geostrophic whereas the cross-front wind is not. The relevant equation set then becomes:

$$v = v_g \tag{5.3}$$

$$\frac{Dv}{Dt} + fu + \frac{\partial \phi}{\partial y} = 0 \tag{5.4}$$

$$\frac{D\theta}{Dt} = 0 \tag{5.5}$$

$$\frac{g\theta}{\theta_0} = \frac{\partial \phi}{\partial z} \tag{5.6}$$

$$\nabla.\underline{u} = 0 \tag{5.7}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}.\nabla$ and note that the full three-dimensional wind field is used for advection.

5.1.3 Cross-frontal circulation

We now aim to find the equation which governs the cross-frontal circulation. This is closely related to the quasi-geostrophic omega equation but is derived using the more accurate equation set. As in the QG theory we start with requiring thermal wind balance to be satisfied:

$$\frac{D}{Dt}\left(f\frac{\partial v}{\partial z}\right) = \frac{D}{Dt}\left(\frac{g}{\theta_0}\frac{\partial \theta}{\partial x}\right) \tag{5.8}$$

Notice that the substantial derivative is being used including the ageostrophic flow advection in the x and z directions . As the wind is geostrophic in the along-front direction we can decompose the wind field via: $\underline{u} = (u_g, v_g, 0) + (u_a, 0, w)$. Taking $\frac{g}{\theta_0}\frac{\partial}{\partial x}$ of equation (5.5) we obtain:

$$\frac{D}{Dt}\left(\frac{g}{\theta_0}\frac{\partial \theta}{\partial x}\right) = Q_1 - \frac{\partial u_a}{\partial x}\frac{g}{\theta_0}\frac{\partial \theta}{\partial x} - \frac{\partial w}{\partial x}\frac{g}{\theta_0}\frac{\partial \theta}{\partial z} \tag{5.9}$$

Notice that unlike QG theory there is no requirement to set the static stability parameter to being independent of the horizontal coordinates. Taking $f\frac{\partial}{\partial z}$ of equation (5.4) we obtain:

$$\frac{D}{Dt}\left(f\frac{\partial v}{\partial z}\right) = -Q_1 - \frac{\partial u_a}{\partial z}f\left(f + \frac{\partial v_g}{\partial x}\right) - \frac{\partial w}{\partial z}\frac{g}{\theta_0}\frac{\partial \theta}{\partial x} \tag{5.10}$$

Using thermal wind balance we can equate the left-hand-sides of equations (5.9) and (5.10) to obtain:

$$\frac{\partial w}{\partial x}\frac{g}{\theta_0}\frac{\partial \theta}{\partial z} - \left(\frac{\partial w}{\partial z} - \frac{\partial u_a}{\partial x}\right)\frac{g}{\theta_0}\frac{\partial \theta}{\partial x} - \frac{\partial u_a}{\partial z}f\left(f + \frac{\partial v_g}{\partial x}\right) = 2Q_1 \tag{5.11}$$

As the front is a two-dimensional structure then the ageostrophic flow in the $x-z$ plane obeys the simple continuity equation:

$$\frac{\partial u_a}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{5.12}$$

Hence we can define an ageostrophic streamfunction, Ψ (not to be confused with the geostrophic streamfunction), by: $u_a = \frac{\partial \Psi}{\partial z}$ and $w = -\frac{\partial \Psi}{\partial x}$. Also define the square of three frequencies: $\mathcal{N}^2 = \frac{g}{\theta_0}\frac{\partial \theta}{\partial z}, \mathcal{S}^2 = \frac{g}{\theta_0}\frac{\partial \theta}{\partial x} = f\frac{\partial v_g}{\partial z}, \mathcal{F}^2 = f\left(f + \frac{\partial v_g}{\partial x}\right) = f\zeta$. Note that ζ is the absolute vorticity of the front. Then equation (5.11) leads to the Sawyer-Eliassen cross-frontal circulation equation:

$$\mathcal{N}^2\frac{\partial^2 \Psi}{\partial x^2} - 2\mathcal{S}^2\frac{\partial^2 \Psi}{\partial x \partial z} + \mathcal{F}^2\frac{\partial^2 \Psi}{\partial z^2} = -2Q_1 \tag{5.13}$$

Note that $\mathcal{N}^2$ is the static stability parameter which here can vary in the horizontal and vertical, $\mathcal{S}^2$ is proportional to the frontal horizontal temperature gradient and $\mathcal{F}^2$ is proportional to the absolute vorticity of the front. Assuming that the coefficients are positive then we can see that $\Psi \propto Q_1$. So if Q_1 is positive then there will be a thermally-direct vertical circulation about the front.

It is instructive to compare equation (5.13) with its quasi-geostrophic counterpart:

$$N^2 \frac{\partial^2 \Psi}{\partial x^2} + f^2 \frac{\partial^2 \Psi}{\partial z^2} = -2Q_1 \tag{5.14}$$

The difference lies in the coefficients where the $\mathcal{S}^2$ term does not appear in the QG system and the other two coefficients are approximated. At a front there are large temperature gradients and vorticity (shear) so that the QG approximation is clearly a poor one. The properties of the Sawyer-Eliassen equation are related to these coefficients. In a mathematical sense the equation is elliptic if $\mathcal{F}^2\mathcal{N}^2 - \mathcal{S}^4 > 0$. If it is elliptic then we can solve the equation if we specify the coefficients everywhere and provide boundary conditions for Ψ. The ellipticity condition turns out to be identical to requiring that $fPV > 0$. This is more obvious if the mixed $x - z$ derivative is removed by making a coordinate transformation. Define new coordinates: $X = x + \frac{v_g}{f}$ and $Z = z$. It is then possible to show that equation (5.13) becomes:

$$\frac{\partial}{\partial X}\left(PV \frac{g\rho}{f\theta_0} \frac{\partial \Psi}{\partial X} \right) + f^2 \frac{\partial^2 \Psi}{\partial Z^2} = -2Q_1 \frac{f^2}{\mathcal{F}^2} \tag{5.15}$$

where for the two-dimensional front $PV = \frac{\theta_0}{fg\rho}(\mathcal{F}^2\mathcal{N}^2 - \mathcal{S}^4)$. The combination of the frontal geostrophy approximation and the coordinate transformation is called the semi-geostrophic theory. It is more accurate than the QG system but has a similar mathematical form in the transformed coordinates; e.g. equation (5.15) and equation (5.14) have a similar form.

It is therefore important to provide a physical picture of the new coordinate system (X, Z). An attraction of these coordinates is that because $u = \frac{Dx}{Dt}$ then equation (5.4) can be written as: $u_g = \frac{DX}{Dt}$. (So as a special case we can see that if $u_g = 0$ then X is conserved.) In addition we note that the slope of X surfaces are given by the expression: $\frac{dz}{dx}\mid_X = -\frac{\mathcal{F}^2}{\mathcal{S}^2}$. For typical values, such as $\xi = 3f$ and $\frac{\partial v}{\partial z} = 10^{-2}s^{-1}$, then $\frac{dz}{dx}\mid_X \approx -0.03$ which is similar to the typical frontal slope. This frontal slope could alternatively be defined as the slope of the θ surfaces: $\frac{dz}{dx}\mid_\theta = -\frac{\mathcal{S}^2}{\mathcal{N}^2}$. Note that θ is also a conserved quantity in principle.

5.1.4 Mechanisms of frontogenesis

A way to quantify the mechanisms of frontogenesis is to look at equation (5.9) governing the changes to the horizontal temperature gradient. It is simplest to remove the sign element of that gradient by multiplying that equation by the temperature gradient itself:

$$\frac{D}{Dt}\left(\frac{1}{2}\left(\frac{\partial b'}{\partial x} \right)^2 \right) = Q_1 \frac{\partial b'}{\partial x} + \frac{\partial w}{\partial z}\left(\frac{\partial b'}{\partial x} \right)^2 - \frac{\partial w}{\partial x}\mathcal{N}^2 \frac{\partial b'}{\partial x} \tag{5.16}$$

We see that there is a geostrophic contribution to frontogenesis which relies on a positive correlation between Q_1 and the horizontal temperature gradient. In addition there is an ageostrophic contribution which gives frontogenesis if there is $\frac{\partial w}{\partial z} > 0$ or a negative correlation between $\frac{\partial w}{\partial x}$ and the horizontal temperature gradient. As regards the geostrophic contribution then there are two simple mechanisms applicable for a two-dimensional front (i.e $\frac{\partial}{\partial y} = 0$):

1. **horizontal deformation:** the simplest form is given by the Bergeron flow $u_g = -\alpha x$ and $v_g = \alpha y$. Then the Q-vector component becomes: $Q_1 = \alpha \frac{\partial b'}{\partial x}$. There is therefore frontogenesis if the deformation rate α is positive.

2. **horizontal shear:** the simplest from is provided by a zonal flow with a constant vertical shear, $u_g = Az$, in thermal wind balance with a uniform along-front temperature gradient. Then $Q_1 = fA \frac{\partial v_g}{\partial x}$. So, for example, there is frontogenesis for cold air lying to the north along the front if there is a positive horizontal shear of the along-front flow.

Therefore the geostrophic flow can act to intensify the front. Also it is responsible for the ageostrophic vertical circulation about the front. In turn this ageostrophic flow can, via the ageostrophic terms in the frontogenesis equation (5.16), act to further increase the frontogenesis. Hence we see that there is a direct link between the process of frontogenesis and the vertical motion at the front. It can be a positive feedback process. Another way to see this positive feedback is to look at the vorticity equation. In the region just ahead of the surface front beneath the zone of ascent there is large vortex stretching. Remembering that the QG scaling is not valid if $\xi \gg f$ then, in this case, the vorticity equation derived from the frontal geostrophy approximation becomes:

$$\frac{D\xi}{Dt} \sim \xi \frac{\partial w}{\partial z} \tag{5.17}$$

This equation shows the existence of a rapid positive feedback wherein as the vorticity increases the stretching increases, as does the vertical motion, leading to a large increase in the vorticity. In fact we can show that the vorticity becomes infinite in a finite time leading to so-called “frontal collapse”.

5.2 Symmetric and convective instabilities

5.2.1 Symmetric instability

The clouds typical of a mid-latitude cyclone can be divided into two categories. There are cloud bands which usually lie approximately parallel to the thermal wind and there are isolated clouds either embedded in these bands or in the form of cellular convection in the cold air behind the cold front. The latter is relatively shallow but can be extremely extensive particularly when the cold air is moving over warmer ocean. In this section we shall be concentrating on the so-called rainbands which can occur in different forms in different positions in the cyclone. Due to their differing geometries and locations they have been categorized into:

- narrow cold-frontal
- wide cold-frontal
- warm sector
- warm frontal
- surge
- postfrontal

The individual structures of these rainbands can be found described in depth in the literature. Here it suffices to say that the common feature of these clouds is that they are oriented parallel with the isentropes, that is along the thermal wind. Given that the baroclinity in mid-latitude cyclones is in the form of quasi-two dimensional fronts then these bands are also essentially two-dimensional with an along front scale of 100's km and an across front scale of 10's km. This across front scale varies from the narrow cold-frontal band or line convection which can be only a few km across to the broader warm sector bands which might be 200 km in width.

It is perhaps important to note that these rainbands can on occasion contain no rain nor be banded! Often they are dominated by ice cloud physics leading to snowfall or at any rate recently melted snow. There may also be essentially only a single cloud line at the front; indeed this is the more common circumstance rather than a series of parallel bands. With these caveats we retain the currently accepted terminology of frontal rainbands.

The aim here is to investigate the essence of the dynamics of these two-dimensional clouds. The first possibility is that these rainbands are mid-latitude squall lines which happen to occur at fronts because there is weak synoptic scale forcing there. However the rainfall rates and totals are much smaller than those typical of such squall lines. Also it is observed that the vertical thermodynamic profile is stable to vertical moist ascent. In other words it appears that they are not convective phenomena. Having said that, embedded within the cloud bands are local regions of enhanced cloudiness and rainfall rate which are probably convective in origin. However the concern here is the main body of the cloud band. Furthermore squall lines are usually oriented with an angle of perhaps 30° from the direction of the thermal wind. They then owe their dynamical structure to the wind shear perpendicular to the line. At a front the bands are usually along the thermal wind and there is extremely strong shear along the band with rather small shear across the band.

A second possibility is that the bands are condensation forced in a stable environment by the frontal ascent which in turn is due to the synoptic scale frontogenesis. This cannot be ruled out but from the above categories of rainbands it is apparent that they can occur away from the frontal zone as well as in the front. For such bands it would seem that some form of spontaneous instability is needed to trigger and sustain the development of the band. The model to be developed here concerns such a dynamical instability known as symmetric instability. In fact the model examines the structure of two-dimensional disturbances parallel to a baroclinic region in the absence of frontogenesis. The role of the frontal forcing in modifying this symmetric instability is a complex question which is the subject of current research. Here we investigate the symmetric disturbances as a general model of two-dimensional motion aligned along a strong baroclinic flow. Observed frontal motion is, in a complicated way, made up from various elements such as symmetric disturbances, cross-frontal circulation due to frontogenesis, and local convective instability. The discussion in the previous section on frontogenesis has highlighted the stable cross-frontal circulations and here we will describe the other components of frontal dynamics just listed.

First we consider the linear theory of conditional symmetric instability. Consider a baroclinic zone oriented parallel to the y-axis. Therefore there is a mean flow in the y-direction, independent of y, which is sheared i.e. $\overline{v} = \overline{v}(x, z)$. By thermal wind balance this flow is consistent with isentropes which have an x and z gradient i.e. which slope in the $x - z$ plane. For a basic state that has potential temperature which is a function of x and z, in thermal wind and hydrostatic balance, the thermodynamic equation is:

$$\frac{\partial b'}{\partial t} + uf\frac{\partial \overline{v}}{\partial z} + w\mathcal{N}^2 = 0 \tag{5.18}$$

where we have used the thermal wind relationship for the basic state, $\frac{\partial \overline{v}}{\partial z} = \frac{g}{f\theta_0}\frac{\partial \overline{\theta}}{\partial x}$ and the basic state static stability is $\mathcal{N}^2 = \frac{g}{\theta_0}\frac{\partial \overline{\theta}}{\partial z}$. The y-component of vorticity, $\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, indicates the circulation in the transverse or (x, z) plane. The linearised equations result from splitting the winds viz; $(u, v) = (0, \overline{v}) + (u', v')$; where a prime indicates a small amplitude perturbation. The linearized vorticity equation can be found by suitable differentiation and subsequent subtraction of x and z momentum equations giving:

$$\frac{\partial \eta}{\partial t} = f\frac{\partial v'}{\partial z} - \frac{g}{\theta_0}\frac{\partial \theta'}{\partial x} \tag{5.19}$$

Thus the evolution of the vorticity is given by the thermal wind imbalance of the perturbation flow. In turn the evolution of this thermal wind imbalance can be found by suitable differentiation and subtraction of the y-momentum equation and equation (5.18) giving:

$$\frac{\partial}{\partial t}\left(f\frac{\partial v'}{\partial z} - \frac{g}{\theta_0}\frac{\partial \theta'}{\partial x}\right) = -\mathcal{N}^2\frac{\partial^2 \Psi}{\partial x^2} + 2\mathcal{S}^2\frac{\partial^2 \Psi}{\partial x \partial z} - \mathcal{F}^2\frac{\partial^2 \Psi}{\partial z^2} \tag{5.20}$$

Using the identity for vorticity in terms of streamfunction, $\eta = \nabla^2\Psi$, and subtracting the time derivative of equation(5.20) from equation (5.19) we get a single equation in the streamfunction:

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial x^2}\right) = -\mathcal{N}^2\frac{\partial^2 \Psi}{\partial x^2} + 2\mathcal{S}^2\frac{\partial^2 \Psi}{\partial x \partial z} - \mathcal{F}^2\frac{\partial^2 \Psi}{\partial z^2} \tag{5.21}$$

The right-hand-side of equation (5.21) is identical to the left-hand-side of the Sawyer-Eliassen equation which is used to determine the cross-frontal ageostrophic circulation. Here this ageostrophic circulation is due to a growing symmetric disturbance whereas in the Sawyer-Eliassen equation it is consistent with geostrophic forcing of frontogenesis. This relationship between the two equations and the role of symmetric instability in frontogenesis will be explored later.

For an unbounded domain the solution of equation (5.21) is the usual simple exponential in space and time: $\Psi = \Psi_0 e^{i(kx+mz)+\sigma t}$ and so the dispersion relation becomes:

$$\sigma = \sqrt{\frac{-\mathcal{F}^2 + 2\mathcal{S}^2\alpha - \mathcal{N}^2\alpha^2}{1+\alpha^2}} \tag{5.22}$$

where $\alpha = \frac{k}{m} = \frac{L_z}{L_x}$ is the aspect ratio of the instability. There will be instability as long as the quantity inside the square root is positive otherwise there will just be stable oscillations. The instability criterion is easiest to see if we make the hydrostatic approximation which is applicable if $L_x \gg L_z$ or $\alpha \ll 1$. This will apply for frontal rainbands which are observed to be say $50km$ wide by a few kilometres deep. Then the instability criterion becomes:

$$-\mathcal{F}^2 + 2\mathcal{S}^2\alpha - \mathcal{N}^2\alpha^2 > 0 \tag{5.23}$$

This inequality must be satisfied if the maximum of the quadratic quantity is positive. This maximum occurs for $\alpha = \frac{\mathcal{S}^2}{\mathcal{N}^2}$ when the maximum is $-\mathcal{F}^2 + \frac{\mathcal{S}^4}{\mathcal{N}^2}$. This is positive provided $\mathcal{F}^2 < \frac{\mathcal{S}^4}{\mathcal{N}^2}$. This instability criterion can be written in terms of a Richardson number $R = \frac{\mathcal{N}^2}{(\frac{\partial \overline{v}}{\partial z})^2}$. It is therefore given by $R < \frac{f}{\zeta}$. If the basic state has a small enough static stability and large enough vertical shear then this criterion can be satisfied. This instability is called symmetric instability. As the moist static stability is much smaller than the dry it is certainly possible that this criterion is satisfied given the large vertical wind shear and high relative humidity in the vicinity of fronts. The resulting motion is referred to as conditional symmetric instability, CSI, as it depends on parcels being lifted to saturation so as to become unstable.

Typical values for a moist front with CSI might be: $\mathcal{N}^2 = 1 \times 10^{-5}s^{-2}$ and $\frac{\partial \overline{v}}{\partial z} = 5\times10^{-3}s^{-1}$ giving a Richardson number of $R = 0.4$. (This can be compared to the Richardson number if the front had been dry, i.e. $\mathcal{N}^2 = 1\times10^{-4}s^{-2}$ giving $R = 4$.) The zone of instability might be in the frontal ascent of the lower troposphere where the relative vorticity is, say, equal to the Coriolis parameter. Thus the front satisfies the criterion for CSI. A typical value for the vorticity is $\zeta = 2f$ giving a maximum growth rate of $\sigma = 0.7 \times 10^{-4}s^{-1}$ or an e-folding time of 4.2 h. The anticyclonic side of the upper jet is even more likely to be unstable by this criterion although we can question whether there would be an adequate supply of moisture there.

There is a short and a long-wave cut-off defined as those aspect ratios for which $\sigma = 0$ and expressions for these wavenumbers are obtained from equation (5.22):

$$\alpha_1 = \frac{f\frac{\partial \overline{v}}{\partial z}}{\mathcal{N}^2}\left(1 + \sqrt{1 - R\frac{\zeta}{f}}\right) \tag{5.24}$$

$$\alpha_2 = \frac{f\frac{\partial \overline{v}}{\partial z}}{\mathcal{N}^2}\left(1 - \sqrt{1 - R\frac{\zeta}{f}}\right) \tag{5.25}$$

There is evidently a "wedge" of instability defined by the limiting aspect ratios in equation (5.25). It is interesting to compare these cut-off aspect ratios with those based on the slope of the basic flow surfaces. These aspect ratios have been given in the previous section on frontogenesis and are repeated here. First define $\alpha_m = -\frac{dz}{dx}\mid_m$ and $\alpha_\theta = -\frac{dz}{dx}\mid_\theta$, giving:

$$\alpha_m = \frac{\mathcal{F}^2}{\mathcal{S}^2} \tag{5.26}$$

$$\alpha_\theta = \frac{\mathcal{S}^2}{\mathcal{N}^2} \tag{5.27}$$

The absolute momentum $m = v + fx$ is proportional to the geostrophic momentum coordinate X $(= \frac{m}{f})$ introduced in the semi-geostrophic theory. It is conserved in this two-dimensional flow. Therefore there are two conserved scalars for this motion, θ and m. The m-surfaces slope with height, as indicated by equation (5.27), if there is vertical shear. The relationship of the slopes of the m and θ-surfaces to the slopes of the surfaces which delimit the wedge of instability is most easily seen from the following:

$$\alpha_1 = \alpha_\theta \left(1 + \sqrt{1 - \frac{\alpha_m}{\alpha_\theta}}\right) \tag{5.28}$$

$$\alpha_2 = \alpha_m \frac{f}{\zeta R} \left(1 - \sqrt{1 - \frac{\alpha_m}{\alpha_\theta}}\right) \tag{5.29}$$

Therefore the wedge of instability is wider than the wedge defined by the m and θ-surfaces. In fact the θ-surface exactly bisects the wedge of instability. The fastest growing wave, in the hydrostatic limit, has $\alpha = \alpha_\theta$. The motion is therefore parallel to the θ surfaces which have a shallow slope typical of the front itself. Another consequence of this orientation of the motion is that parcels ascend with nearly zero buoyancy and therefore small pressure perturbations. Symmetric instability takes the form of so-called rolls lying parallel to the front. Given the typical slope of frontal isentropes and a depth of frontal rainband layer we might expect $L_x \approx 330km$ and $L_z \approx 3km$ to be characteristic scales. Given this geometry a typical ascent and precipitation rate might be $20cms^{-1}$ and $10mmh^{-1}$ respectively.

As both m and θ are conserved we can give a rather elegant simplified picture of the instability using what is known as parcel theory. Imagine lifting a parcel of air from an initial location to a point within the wedge of instability. For the CSI unstable case with θ-surfaces steeper than the m-surfaces the parcel will find itself in a region of lower θ and higher m. It will thus have a vertical force accelerating it towards its "home" θ surface and a horizontal force towards its home m surface. The resultant force is directed away from the parcel's initial location; in the stable case with m-surfaces steeper than θ-surfaces then it is towards the original position. Notice, though, that there is no requirement for this resultant force to be directed along the parcel displacement vector.

Mesoscale observations of fronts indicate that it is not uncommon for the front to have m-surfaces parallel to the θ_e-surfaces in regions of saturated ascent. This is the condition for neutral stability to CSI indicating that frontal rainbands are likely to be a consequence of CSI.

5.2.2 Convective instability

A special case of the analysis just made for symmetric instability is when the effects of the Earth's rotation can be neglected. This will be true for small space and time scale phenomena. This limit can be obtained by setting $f = 0$ in equation (5.22) to give:

$$\sigma = \sqrt{-\frac{\mathcal{N}^2\alpha^2}{1+\alpha^2}} \tag{5.30}$$

We then obtain the convective instability criterion that $\mathcal{N}^2 < 0$. This is satisfied if either θ decreases with height for an unsaturated flow or if θ_e decreases with height for a saturated flow. The maximum growth rate occurs for $\alpha \to \infty$ or $L_x \to 0$. Thus the maximum growth in convective instability occurs for very narrow zones of ascending air parcels. This is sometimes referred to as nailbed convection. It accords with observations of cumuloform clouds which are composed of narrow plumes or thermals. It does not account for the larger scale organization of the whole cloud itself. It is thought that this organization is due to the wind shear in which the cloud exists. This will be discussed in the next section.

5.3 Severe convective storms

5.3.1 Parcel model

A simple and practically useful model of convection in the atmosphere is called the parcel model. Its basic assumption is that we can divide the atmosphere into a single parcel, of undetermined size, and its environment. The parcel moves due to forces acting on it from the environment but the model neglects the effect of the parcel back on its environment. This can be done if we ignore the perturbation pressure field due to the parcel to give the vertical momentum equation as:

$$\frac{Dw}{Dt} = b' \tag{5.31}$$

We can multiply this equation by the vertical motion itself to obtain an energy equation. In addition it is possible to write the right-hand-side as the derivative of an integral, viz:

$$\begin{aligned} \frac{D}{Dt}\frac{1}{2}w^2 &= b'w && (5.32)\\ &= \frac{D}{Dt}\int_{z_0}^{z} b'dz && (5.33) \end{aligned}$$

where z_0 is the initial location of the parcel. Note that the derivative means "following a parcel" and the prime means "parcel minus environment". We can then obtain a conservation type energy equation:

$$\frac{D}{Dt}\left(\frac{1}{2}w^2 - CAPE\right) = 0 \tag{5.34}$$

where the convective available potential energy, $CAPE$, is defined as: $CAPE = \int_{z_0}^{z} b' dz$.

It can be shown that the $CAPE$ is equal to the so-called positive area on a tephigram. It indicates the energy available for convection. As an example if a parcel was warmer than its surroundings by an average of $3°C$ over a vertical traverse through a cloud of $9km$ then $CAPE = 900 Jkg^{-1}$. Equation (5.34) shows that if the parcel had no vertical kinetic energy at cloud base it would be ascending at $42ms^{-1}$ at cloud top. This is clearly a large ascent rate and shows how powerful convection is in the Earth's atmosphere.

Another key aspect of thunderstorm thermodynamics is the fact that there is a rather universal profile of θ_w or θ_e of the air which feeds the storm. It has a mid-level minimum in θ_w showing that a lifted parcel from the boundary layer can ascend to near to the tropopause. As rain falls from the storm it can evaporate into sub-saturated air to maintain a near saturated downdraught. This means that θ_w is conserved by such downdraught parcels. They can only be negatively buoyant if they descend from **below** the mid-level minimum in θ_w. Hence downdraughts are usually fed from around 600 or $700mb$.

5.3.2 Types of convective storm

As well as the characteristic thermodynamic structure severe convective storms exist in characteristic environmental wind profiles. In fact we can classify storm types in terms of these wind profiles. The two factors of wind shear and $CAPE$ can be incorporated into a single non-dimensional factor $R = \frac{CAPE}{\frac{1}{2}(\Delta U)^2}$ where ΔU is the wind shear in the lowest $6km$ of the atmosphere. There are basically three types of such convective storm: multicells, supercells and squall lines. There are however various sub-divisions within these categories:

- **Multicells**: these are composed of a sequence of individual cells which grow and decay. As one cell decays another cell is produced which enables the storm to persist for several hours. They exist in a wind profile which has a strong directional wind shear but with perhaps only a moderate speed shear. The factor R lies in the range $30 < R < 300$.

    1. *propagating storm*: this covers the more usual case where the multicellular storm propagates with a speed and direction different to that of the wind at any level.
    2. *stationary storm*: in such systems the regeneration occurs over a single location so that although individual cells move away from their generation region the storm stays over one geographical location.
    3. *splitting storm*: sometimes an individual cell can be seen to divide into 2 cells which propagate in somewhat different directions.

- **Supercells**: supercell storms are generally composed of a single cloud, or cell, which persists in a nearly steady state for several hours. The supercell is also highly three-dimensional and exists in a wind profile which has large directional and speed shear. The factor R lies in the range $0 < R < 50$.

    1. *right-moving*: this propagates in a direction lying to the right of the wind direction at any level and it is the most common supercell storm type.

2. *left-moving*: this propagates to the left of the wind direction and is relatively unusual.

- **Squall lines**: these are quasi-two-dimensional being several hundred kilometres long and perhaps $50km$ across. Generally they are oriented approximately parallel to the upper tropospheric flow. They have a relatively narrow zone of convection with a long trailing anvil or stratiform region which also produces precipitation.

  1. *mid-latitude*: usually the wind component perpendicular to the line has a large low-level shear with little flow at upper levels
  2. *tropical*: these exist in a wind profile which exhibits a low-level easterly jet and a separate upper-level jet.

In addition there are storms initiated over mountains, orographic convection, and flash floods which relate to severe storms with consequent surface flooding.

5.3.3 Updraught-downdraught feedback

It is clear from this discussion of storm types that wind shear plays a significant role in maintaining long-lasting convective storms. Here we try to suggest why this might be the case. Storms are generally composed of an updraught of warm moist air from the boundary layer which rises up in the storm to exit in the upper-troposphere as a nearly horizontal anvil cloud. The storm can be thought of as drawing in boundary layer air and depositing it into the upper troposphere. The rain which is thus generated falls and due to drag and evaporation a downdraught is created which is negatively buoyant. This originates at mid-levels and descends to flow out at the surface as a so-called "density current".

For this to be maintained it is essential that the rain falls out of the updraught into ambient unsaturated air and that the cold density current does not prevent the ascent of air into the updraught. From a geometrical viewpoint this favourable situation can be produced if the updraught leans at an angle to the vertical and the density current which flows towards the approaching warm air remains stationary relative to the storm. The interface between the cold air from the downdraught and the incoming warm air at the ground is called the gust front. The propagation of the gust front ahead of the storm is prevented if there is a strong low-level flow inhibiting its progress. Thus we conclude that a long-lived storm requires a wind shear between low-levels and mid-levels.

5.3.4 Tornadogenesis

Tornadoes are small-scale funnels of rapidly rotating air which exist between the cloud base and the surface. To explain their formation we must explain how the air in the thunderstorm can acquire spin about a vertical axis. It is thought that this process is independent of the Earth's rotation and so we will neglect that in the following discussion. The air approaching the storm has horizontal vorticity due to its vertical shear. As this air is lifted into the updraught there is tilting of this horizontal vorticity into the vertical giving a dipole of cyclonic and anticyclonic vorticity. Vortex stretching will amplify this dipole.

This process is thought to explain the relatively modest spin in the mid-cloud level which is observed in these storms and which leads to the so-called mesocyclone. However to produce the much larger tornadic rotation another process is needed. This is thought to be due to the increase in the horizontal component of vorticity as the air reaches the gust front. At the gust front there is baroclinic generation of horizontal vorticity due to the large horizontal temperature gradient there. When this is tipped into the vertical and is stretched extreme rotation can be produced. This explains why the gust front is a preferred location for the generation of tornadoes.

5.4 Orographic processes

5.4.1 Effects of mountains on the atmosphere

There are a multitude of ways in which mountains affect the atmosphere. Here we focus on those that occur on the mesoscale. The following is a list of some of these effects:

1. **Diverts air over and around the mountain**: the flow can be diverted partly over the mountain, leading to vertically propagating mountain waves, and partly around the mountain, leading to horizontal flow structures such as lee vortices, vortex shedding and larger-scale Rossby waves.

2. **Exerts drag on the atmosphere**: the flow creates a surface pressure perturbation as the air moves over and around the mountain. This pressure field represents a force exerted by the air on the mountain and an equal but opposite force exerted by the mountain on the atmosphere. This is referred to as surface drag. Part of the mountain flow is in the form of vertically propagating mountain waves. These waves can "break" at upper levels in the atmosphere and then exert a drag at the breaking level.

3. **Creates local wind regimes**: depending on the specific shape of the orography there are a variety of local names for characteristic wind regimes. These include: fóhn, bora, mistral, etc. There are also names for upslope and downslope flows i.e. anabatic and katabatic winds and downslope windstorms. Valley winds and flow regimes are also of interest.

4. **Acts as an elevated heat source**: due to the solar heating of the mountain slopes the orography acts as an elevated heat source. For example the Tibetan plateau is a major elevated heat source having large implications for the global as well as the local circulation.

5.4.2 Mountain wave equation

To examine flow over mountains it is possible to construct a simplified linear model for flow over a set of two-dimensional ridges oriented along the y-direction. The model allows us to predict the flow, pressure and temperature in terms of the basic state flow and static stability. For simplicity we will take a sufficiently narrow mountain range such that the Coriolis force can be neglected. (The model can easily be extended to include the effects of the Earth's rotation.) We will assume that the flow is steady so that $\frac{\partial}{\partial t} = 0$ and that there is a basic state wind, $\overline{U}$, which is a function of height only. So the basic state flow is impinging on the mountain normal to the ridge. The Boussinesq equations with these assumptions having been made are:

$$\overline{U}\frac{\partial u'}{\partial x} + w'\frac{d\overline{U}}{dz} + \frac{\partial \phi'}{\partial x} = 0 \tag{5.35}$$

$$\overline{U}\frac{\partial w'}{\partial x} + \frac{\partial \phi'}{\partial z} = b' \tag{5.36}$$

$$\overline{U}\frac{\partial b'}{\partial x} + w'N^2 = 0 \tag{5.37}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \tag{5.38}$$

where a prime indicates the mountain induced flow which has been taken to be of small amplitude thereby allowing a linearization of the equations. It is usual to manipulate thse linear equations into a single equation for one quantity, w', the vertical velocity. Substituting equation (5.38) into equation (5.35) and differentiating the resulting equation with respect to height we obtain:

$$-\overline{U}\frac{\partial^2 w'}{\partial z^2} + w'\frac{d^2\overline{U}}{dz^2} + \frac{\partial^2 \phi'}{\partial x \partial z} = 0 \tag{5.39}$$

To remove the pressure gradient term we can form the horizontal derivative of equation (5.36):

$$\overline{U}\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial x \partial z} = \frac{\partial b'}{\partial x} \tag{5.40}$$

Finally we substitute into equation (5.40) for the horizontal temperature gradient from equation (5.37) and then subtract the resulting equation from equation (5.39) to give:

$$\frac{\partial^2 w'}{\partial x^2} + \frac{\partial^2 w'}{\partial z^2} + \left(\frac{N^2}{\overline{U}^2} - \frac{1}{\overline{U}}\frac{d^2\overline{U}}{dz^2}\right) w' = 0 \tag{5.41}$$

This is a wave equation describing the structure of linear mountain waves. The behaviour of its solution is governed by the parameter:

$$l^2 = \frac{N^2}{\overline{U}^2} - \frac{1}{\overline{U}}\frac{d^2\overline{U}}{dz^2} \tag{5.42}$$

and l^2 is called the Scorer parameter. Whilst this equation has a rather general validity we are going to look at a particularly simple mountain profile: $h(x) = h_m cos(kx)$, with a uniform basic state wind and static stability. Hence the Scorer parameter is constant, $l^2 = \frac{N^2}{\overline{U}^2}$. Solutions to equation (5.41) are:

$$w'(x, z) = w_1(z)cos(kx) + w_2(z)sin(kx) \tag{5.43}$$

If we substitute equation (5.43) into the wave equation (5.41) then $w_i = (w_1, w_2)$ both satisfy:

$$\frac{d^2 w_i}{dz^2} + \left(l^2 - k^2\right) w_i = 0 \tag{5.44}$$

Solutions to equation (5.44) are of the form:

$$w_i = \begin{cases} A_i e^{\mu z} + B_i e^{-\mu z} & \text{if } k > l \\ C_i cos(mz) + D_i sin(mz) & \text{if } k < l \end{cases} \tag{5.45}$$

where A_i, B_i, C_i, D_i are constants to be determined by appropriate boundary conditions, $m = \sqrt{l^2 - k^2}$ and $\mu = \sqrt{k^2 - l^2}$. The two cases correspond to short waves ($k > l$ or $\lambda < 2\pi\frac{\overline{U}}{N}$) and long waves ($k < l$ or $\lambda > 2\pi\frac{\overline{U}}{N}$). For typical values of, say, $\overline{U} = 10ms^{-1}$ and $N = 10^{-2}s^{-1}$ then the division between these two regimes is for a mountain wavelength $\lambda \approx 6km$. At the ground we can use the condition that $w(x, 0) = \overline{U}\frac{\partial h}{\partial x} = -\overline{U}h_m k sin(kx)$. This is a linearisation of the kinematic condition that the air must flow parallel to the mountain surface at the ground. At heights well above the mountain we require another boundary condition. Consider the two cases:

1. **Short waves** $k > l$: the solution is composed of a part exponentially decreasing with height and a part exponentially increasing with height. The latter is unphysical so we set $A_i = 0$. Using the boundary condition at the ground ($z = 0$) gives: $B_1 = 0$ and $B_2 = -\overline{U}h_m k$. Hence the solution in this case is:

$$w'(x, z) = -\overline{U}h_m k sin(kx)e^{-\mu z} \tag{5.46}$$

2. **Long waves** $k < l$: using trigonometric addition formulae the vertical velocity can be written in this case as:

$$w'(x, z) = E_1 sin(kx+mz)+E_2 sin(kx-mz)+E_3 cos(kx+mz)+E_4 cos(kx-mz) \tag{5.47}$$

The solution is in the form of components which tilt upstream with height (with argument $kx + mz$) and components that tilt downstream with height (with argument $kx - mz$). At the ground, $z = 0$, we impose the boundary condition which gives $E_3 + E_4 = 0$ and $E_1 + E_2 = -\overline{U}h_m k$. The condition for large heights is not so easy to see as none of the components decay or increase with height as for the other case. In fact it can be shown that the upstream tilted wave transports energy upwards whereas the downstream component has energy propagating down from infinity. Therefore, somewhat contrary to one's naive expectation, the appropriate condition is to remove the components which tilt downstream, i.e. $E_2 = E_4 = 0$. Therefore the solution in this case is given by:

$$w'(x, z) = -\overline{U}h_m k sin(kx + mz) \tag{5.48}$$

> One consequence of this upstream tilted solution is that the approaching airstream "feels" the presence of the mountain before it actually arrives over the mountain. So, for example, there is ascent upstream of the mountain immediately above the (flat) ground. This leads to the idea that, in the presence of the Earth's rotation, vortex stretching takes place upstream of the mountain deflecting the air to the north. Subsequently there is vortex compression over the mountain itself and the air is deflected south.

It is worth noting that for isolated mountains of width a then the parameter $\frac{1}{a}$ plays the role of k in the above theory. The additional possibility now arises that the air can go around as well as over the mountain. The parameter which determines the extent to which the air goes around or over is the Froude number given by the expression: $F = \frac{\overline{U}}{Nh_m}$. This parameter is essentially the ratio of the vertical wave scale and the height of the mountain. For low mountains $F \gg 1$ and the air flows predominantly over the mountain whereas for high mountains, i.e. $F \ll 1$, the air flows predominantly around the mountain. In this latter case linear theory cannot be used. Numerical solutions show that such low Froude number flow results in the production of lee vortices and vortex shedding.

5.4.3 Wave drag and momentum flux

It is easy to see using the linear model of mountain flow that there will in general be a pressure difference across the mountain at the surface. An expression for this can be obtained from the x-momentum equation (5.35) which using the continuity equation can be written in the form: $\frac{\partial \phi'}{\partial x} = \overline{U}\frac{\partial w'}{\partial z}$. Substituting in the solutions given above for the vertical velocity gives the following expressions for the pressure field at the surface:

$$\phi'_s = \begin{cases} -\overline{U}^2 \mu h_m cos(kx) & \text{if } k > l \\ -\overline{U}^2 m h_m sin(kx) & \text{if } k < l \end{cases} \tag{5.49}$$

Thus we see that in the case of short waves there is low pressure at the crest of the ridge. In contrast for longer waves there is high pressure on the upstream side and low pressure on the downstream side of the ridge. The surface pressure field causes a force on the mountain exerted by the atmosphere and an equal but opposite force on the air exerted by the mountain. This force on the atmosphere, D_s, is given by the expression:

$$D_s = -\int_{-\infty}^{\infty} p'_s \frac{\partial h}{\partial x} dx \tag{5.50}$$

It is easy to show that for short waves $D_s = 0$ and for long waves $D_s < 0$. As this force is negative it is referred to as a drag.

In addition to this surface pressure drag, in the case of $k < l$, there is a vertical flux of momentum carried by the waves as there is non-zero correlation of u' and w'. This momentum flux causes a drag on the atmosphere above the mountain in any region where the flux has a vertical gradient. These regions correspond to where wave breaking is occuring. Such regions cannot be described with linear theory.

5.4.4 Lee waves

We have seen how depending on the horizontal scale of the mountain compared to the Scorer parameter there will be either an evanescent decrease of wave amplitude with height or an upward vertical propagation. This leads to the possibility of a situation occurring in which there is a layer just above the mountain in which there is vertical propagation capped by a layer in which the waves are trapped. Then the wave energy, instead of propagating to infinite heights, will be forced to propagate horizontally. This leads to the formation of lee waves stretching downstream of the mountain. It is easy to see that the general criterion for this to happen is that the Scorer parameter must decrease with height in such a way that: $l_U < k < l_L$ where suffices L and U refer to the lower and upper layers of the atmosphere. This can happen if the wind or static stability changes speed with height. Such lee waves are evident as lenticular clouds on satellite pictures or from visual observations from the ground.

Chapter 6

Problem Sheets

6.1 Problem Sheet 1: Vorticity & Potential Vorticity

1. Use the definition of geostrophic vorticity to prove the following form of the thermal wind equation ((2.25) in the notes):

$$\frac{\partial \xi_g}{\partial z} = \frac{1}{f_0} {\nabla_h}^2 b' \tag{6.1}$$

2. If there is ascent at a height of 4 km of $15cms^{-1}$ evaluate the rate of change of absolute vorticity following the geostrophic flow typical of the lower troposphere. What is the rate of change of temperature at 4 km?

3. At a height of about 5 km the relative vorticity at 52°N is increasing at a rate of $5 \times 10^{-6} s^{-1}/3h$. The wind is a 50 knot south-westerly and the vorticity decreases towards the northeast at a rate of $4 \times 10^{-6} s^{-1}/100km$. Sketch on a suitable diagram contours of relative vorticity and some wind vectors. Use the QG vorticity equation to estimate the horizontal divergence (DIV) on a β-plane. You can assume that $DIV = -\frac{\partial w}{\partial z}$.

4. The geostrophic streamfunction is given by the following expression:

$$\psi = c \left\{ y \left(cos\left(\pi \frac{z}{H}\right) - 1 \right) + \frac{sin(k(x - ct))}{k} \right\} \tag{6.2}$$

   where c is a constant speed, $H = 10km$, and k is a zonal wavenumber.

   (a) Assuming $\beta = 0$ use the quasi-geostrophic vorticity equation to find the horizontal divergence ($= -\frac{\partial w}{\partial z}$) consistent with ψ.

   (b) If $w(z = 0) = 0$ obtain an expression for $w(x, y, z, t)$ by integrating the continuity equation with respect to height.

   (c) Sketch the ψ-field at $10km$ and at the surface. Indicate regions of maximum divergence and convergence and positive and negative vorticity advection.

(d) Obtain an expression for w directly from the thermodynamic equation using the expression for ψ. Which value of k makes this expression the same as that in (b) from the vorticity equation?

5. Using thermal wind balance prove equation (2.33) in section 2.4.1 of the notes. This is the vital step in demonstrating the conservation of quasi-geostrophic potential vorticity.

6. An old tropical cyclone has re-curved into the mid-latitude westerly belt. It has associated with it a maximum temperature anomaly at 300 mb (about 9 km) of 30K and a height anomaly at this level of 34 decametres; these anomalies having a characteristic horizontal scale of 750 km. Use the hydrostatic equation to give an estimate of the equivalent pressure anomaly at 9 km. Estimate: the relative geostrophic vorticity at this level and the vertical gradient of vorticity (using equation (6.1)). Use these estimates to find the relative vorticity at the surface.

   Suppose the buoyancy anomaly increases with height from zero at the surface to the value at 300 mb given above. Estimate the QG potential vorticity typical of the cyclone at mid-levels (5 km).

7. Obtain expressions for the potential temperature and v geostrophic wind component for the uniform ball QG potential vorticity solution given in the notes. Evaluate the maximum wind if $\hat{q} = 2.5 f_0$, $r_0 = 250km$, $N^2 = 1 \times 10^{-4} s^{-2}$, and $f = 1 \times 10^{-4} s^{-1}$.

6.2 Problem Sheet 2: Vertical Motion

1. Suppose that in a jet entrance region at a height of $10km$ the westerly wind increases from $25ms^{-1}$ to $50ms^{-1}$ over a horizontal distance of $1000km$. The surface wind is zero. Estimate the magnitude and direction of the Q-vector. assuming a meridional (ie. cross-jet direction) length scale of $500km$ estimate the maximum vertical velocity.

2. Consider the Bergeron deformation flow, $u = -\alpha x$ and $v = \alpha y$. Show that in this case the omega equation can be written in the form:

$$N^2 \frac{\partial^2 w}{\partial x^2} + {f_0}^2 \frac{\partial^2 w}{\partial z^2} = 2 \frac{\partial Q_1}{\partial x} \tag{6.3}$$

   where:

$$Q_1 = \alpha \frac{\partial b'}{\partial x} \tag{6.4}$$

   Suppose that at $700mb$ the buoyancy distribution is given by the equation: $b' = b_1 tan^{-1}\left(\frac{x}{L_1}\right)$ where $b_1 = \frac{g20K}{\pi\theta_0}$ and $L_1 = 200km$. Find expressions for Q_1 and $\frac{\partial Q_1}{\partial x}$. At which value of x is the maximum in Q_1 and $\frac{\partial Q_1}{\partial x}$? Sketch the horizontal variation of these quantities and b'.

   Make a suitable scaling approximation for the second derivatives on the left-hand-side of the omega equation to find the maximum vertical motion that might be expected if $\alpha = 10^{-5}s^{-1}$.

3. For a certain mid-latitude weather system with horizontal and vertical scales of $500km$ and $5km$ respectively, a region of ascent with peak magnitude of $10cms^{-1}$ is observed. Estimate the magnitude of VA and TA, assuming A and B, in the forcing term of the omega equation, are of the same sign and magnitude. Hence, by making a plausible assumption about the variation of the vertical motion with height, and assuming that $VA > 0$, find the change in relative vorticity over a fixed point in 12 hours.

4. In a case with zero mean flow determine the frequency of inertia-gravity waves with the following scales and calculate how far energy would travel in 1 day:

   (a) $L_x = 30km$ and $L_z = 30km$
   (b) $L_x = 30km$ and $L_z = 1km$
   (c) $L_x = 30km$ and $L_z = 100m$
   (d) $L_x = 5km$ and $L_z = 2km$

   Which meteorological phenomena might be responsible for forcing the waves described in examples (a) and (d)?

5. In the case of zero mean flow sketch a graph of frequency versus aspect ratio for inertia-gravity waves in the following cases:

(a) non-hydrostatic with rotation

(b) hydrostatic with rotation

(c) non-hydrostatic without rotation ($f = 0$)

(d) hydrostatic without rotation

Deduce the conditions under which waves are only weakly affected by either making the hydrostatic approximation or neglecting rotation.

6.3 Problem Sheet 3: Waves and Instabilities

1. Imagine that at the tropopause there is a local region of cold air. Using concepts drawn from potential vorticity invertibility, and shown schematically in Figure 4.2, and the thermodynamic equation show diagrammatically that you would expect a westward phase tilt with height of pressure for a baroclinic wave.

2. A $2000km$ baroclinic edge wave located at the ground has an amplitude there of $5K$.

   (a) determine the e-folding height for the wave, ie: H_R.

   (b) what is the reduction in amplitude at $10km$

   (c) evaluate the magnitudes of the buoyancy, pressure, meridional wind and relative vorticity amplitudes at the ground

   (d) determine the phase speed of the wave if the westerly wind:

      i. increases from 0 to $30ms^{-1}$ in $10km$

      ii. increases from 5 to $35ms^{-1}$ in $10km$

3. One model of observed polar lows is that of an Eady wave in a baroclinic flow capped by an effective lid at $2km$. Taking typical values, evaluate the wavelength and growth rate of the most unstable mode, indicated by this model.

4. (a) Show that the amplitude of the streamfunction corresponding to the potential vorticity perturbation $q' = Ae^{ik(x-\alpha z)}$ in an unbounded Eady basic state, with zonal wind shear A, increases by a factor of 5 as the slope (α^{-1}) changes from $-\frac{f}{2N}$ to vertical

   (b) if the basic state shear is $3ms^{-1}$ per km how long does this take?

   (c) defining the streamfunction growth rate to be:

   $$\sigma = \frac{1}{|\psi|}\frac{d|\psi|}{dt}$$

   show that:

   $$\sigma = -\frac{2a}{1+a^2}\frac{fA}{N}$$

   where $a = \alpha\frac{f}{N}$.

6.4 Problem Sheet 4: Mesoscale Weather Systems

1. The growth rate equation for symmetric instability, equation (5.22), has a long wave (α_2), and a short wave, (α_1), cut-off such that there is growth only in the range: $\alpha_2 < \alpha < \alpha_1$. Show that the expressions for these cut-off values are as given in equation (5.24) and (5.25).

2. For a certain frontal rainband the following data have been collected at $50°N$: $\mathcal{N}^2 = 2\times10^{-5}s^{-2}$, relative vorticity of $1\times10^{-4}s^{-1}$, and vertical wind shear of $12ms^{-1}(km)^{-1}$. If the vertical wavelength can be taken to be $1.5km$ find the horizontal wavelength and e-folding time for the fastest growing symmetrically unstable disturbance. Assume that the hydrostatic approximation can be made.

3. For the frontal rainband environment we showed that the instability criterion for symmetric instability was $R < \frac{f}{\zeta}$. The (Ertel-Rossby) potential vorticity of the basic state is given by the expression: $PV = \frac{1}{\rho}\left(\zeta\frac{\partial\theta}{\partial z} - \frac{\partial v}{\partial z}\frac{\partial\theta}{\partial x}\right)$. Show that the symmetric instability criterion is equivalent to $fPV < 0$. Note that generally in the atmosphere, apart from in special areas like fronts, then the potential vorticity has the same sign as the Coriolis parameter.

4. Suppose that a new study of the dynamics of supercell convective storms suggested that they occur if R $(= \frac{CAPE}{\frac{1}{2}(\Delta U)^2})$ is equal to 5, where ΔU is the wind difference over the lowest 6km of the atmosphere. If a typical updraught parcel has an excess temperature compared to its surroundings of $3.8K$ and the storm is observed to be $12km$ in depth, then estimate the shear of the wind. Why is it thought that wind shear is advantageous for the production of severe local storms?

5. A certain mountain range has an equivalent horizontal wavelength of $12km$. For an upstream flow with $\overline{U} = 8ms^{-1}$ and $N^2 = 1.4\times10^{-4}s^{-2}$ find the vertical wavelength of linear mountain waves. If the mountain has a height of $800m$ find the maximum vertical velocity at the surface and its location relative to the mountain profile. Find the magnitude of the maximum surface pressure perturbation, in millibars, caused by the mountain.

6. For a mountain range with an effective horizontal wavelength (λ) of $7km$ there is an upstream flow of $6ms^{-1}$ which is constant with height. In the lower troposphere find the vertical gradient of potential temperature which gives a Scorer parameter which satisfies the expression: $l = \frac{4\pi}{\lambda}$. Show that the minimum possible vertical gradient of θ in the upper troposphere, if lee waves are observed in the lower troposphere, is exactly one quarter of the value in the lower troposphere.