Some calculations using the gradient wind equation

ATM 210 – Fall, 2023 – Fovell

In our "go with the flow" coordinate system (see figure below), the gradient wind speed V is non-negative, and the $\hat{n}$ direction is positive to the left of the flow. We can write the gradient wind equation as:

$$\frac{V^2}{R} + fV = -\frac{1}{\rho}\frac{dp}{dn}, \tag{1}$$

where R is the radius of curvature ($R > 0$ CCW), f is the Coriolis parameter ($= 2\Omega \sin\phi$, ≥ 0 in the Northern Hemisphere), ρ is density, p is pressure, Ω is Earth's angular velocity, and ϕ is latitude. The terms represent centripetal (or centrifugal), Coriolis, and pressure gradient accelerations, respectively. Note $\frac{dp}{dn} = \frac{\Delta p}{\Delta n}$, and is the change of pressure in the $+\hat{n}$ direction. If pressure drops in the $+\hat{n}$ direction, $\frac{dp}{dn} < 0$.

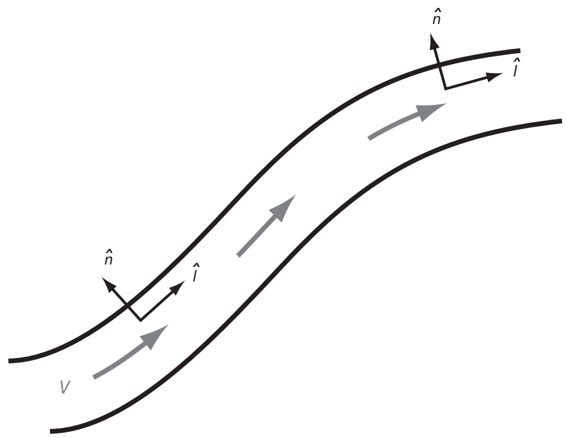

Figure 1: Our coordinate system, defined by $\hat{l}$ (along the flow) and $\hat{n}$ (positive to left).

If $R = \infty$, we can replace V with the geostrophic wind speed V_g and utilize that to rewrite the gradient wind equation in terms of V_g as

$$\frac{V^2}{R} + f(V - V_g) = 0. \tag{2}$$

The latter form shows that $V \neq V_g$ if there is non-infinite curvature ($R < \infty$).

Geostrophic wind speed

Let's do some example calculations. Suppose the pressure difference Δp = -10 mb = -1000 N/m^2 over a distance of Δn = 1000 km = 10^6 m, for air near sea-level (ρ = 1 kg/m^3) at 45°N ($f = 10^{-4}$ 1/s). Further suppose the isobars are not curved, so $R = \infty$. That is, the wind is geostrophic. We can solve (1) for the wind speed V, call it V_g, and find it is 10 m/s.

$$\begin{aligned} V_g &= -\frac{1}{f\rho}\frac{\Delta p}{\Delta n} \\ &= -\frac{1}{(10^{-4}\ 1/\mathrm{s})(1\ \mathrm{kg/m^3})}\frac{-1000\ \mathrm{N/m^2}}{10^6\ \mathrm{m}} \\ &= +10\ \mathrm{m/s} \end{aligned}$$

Gradient wind speed

Suppose now we make the air curve CCW with a radius of curvature of +1000 km (+10^6 m). We are not changing the pressure gradient or latitude or density, so V_g is not affected. We can use equation (2) now that we know what V_g is but we have to solve the quadratic:

$$\begin{aligned} V &= -\frac{fR}{2} \pm \left[\frac{f^2R^2}{4} + fRV_g\right]^{1/2} \\ &= -\frac{(10^{-4}\ 1/\mathrm{s})(10^6\ \mathrm{m})}{2} \pm \left[\frac{(10^{-4}\ 1/\mathrm{s})^2(10^6\ \mathrm{m})^2}{4} + (10^{-4}\ 1/\mathrm{s})(10^6\ \mathrm{m})(10\ \mathrm{m/s})\right]^{1/2} \\ &= -50\ \mathrm{m/s} \pm \left[2500\ \mathrm{m^2/s^2} + 1000\ \mathrm{m^2/s^2}\right]^{1/2} \\ &= -50\ \mathrm{m/s} \pm \left[3500\ \mathrm{m^2/s^2}\right]^{1/2} \\ &= -50\ \mathrm{m/s} \pm 59\ \mathrm{m/s} \\ &= +9\ \mathrm{m/s}\ \text{or}\ -109\ \mathrm{m/s} \end{aligned}$$

The negative root of the quadratic results in $V < 0$, which is unphysical. So with this CCW radius of curvature of this magnitude, the gradient wind is 10% slower than the geostrophic wind V_g. The gradient wind is subgeostrophic.

Making the air curve CW instead at the same radius of curvature (R = -10^6) makes the gradient wind speed supergeostrophic.

$$
\begin{aligned}
V &= -\frac{fR}{2} \pm \left[\frac{f^2R^2}{4} + fRV_g\right]^{1/2} \\
&= -\frac{(10^{-4}\ 1/\mathrm{s})(-10^6\ \mathrm{m})}{2} \pm \left[\frac{(10^{-4}\ 1/\mathrm{s})^2(-10^6\ \mathrm{m})^2}{4} + (10^{-4}\ 1/\mathrm{s})(-10^6\ \mathrm{m})(10\ \mathrm{m/s})\right]^{1/2} \\
&= +50\ \mathrm{m/s} \pm \left[1500\ \mathrm{m^2/s^2}\right]^{1/2} \\
&= +50\ \mathrm{m/s} \pm 39\ \mathrm{m/s} \\
&= +11\ \mathrm{m/s}\ \text{or}\ +89\ \mathrm{m/s}
\end{aligned}
$$

In this case, we get two positive wind speeds, but one is absurd. The other demonstrates that making the air curve CW increases the wind speed by 10% above the geostrophic value.

Cyclostrophic wind speed

Cyclostrophic flow is valid at small scales in which the Coriolis force is negligible. We get the equation for cyclostrophic flow by taking (1) and ignoring the fV term. So, using the same values as above, but excluding Coriolis, we end up with

$$
\begin{aligned}
\frac{V^2}{R} &= -\frac{1}{\rho}\frac{\Delta p}{\Delta n} \\
V &= \left[-(R)\frac{1}{\rho}\frac{\Delta p}{\Delta n}\right]^{1/2} \\
&= \left[-(10^6\ \mathrm{m})(-0.001\ \mathrm{m/s^2})\right]^{1/2} \\
&= \left[1000\ \mathrm{m^2/s^2}\right]^{1/2} \\
&= +32\ \mathrm{m/s}
\end{aligned}
$$

This cyclostrophic wind speed is much faster than the geostrophic and gradient winds using the same pressure gradient, density, horizontal scale, and radius of curvature. It demonstrates cyclostrophic winds can be very fast.

However, the example is not realistic because Coriolis is non-negligible over such large horizontal scales as 10^6 m or 1000 km. But, suppose we have the same pressure drop (-10 mb) over 1 km instead. Now $\Delta n = R = 1000$ m. Prove it to yourself you will get the same cyclostrophic wind! And it doesn't matter whether you spin it CCW or CW ($R < 0$ or $R > 0$). As R changes sign, so does $\frac{\Delta p}{\Delta n}$.

Tornadoes represent intense, small scale circulations. The largest recorded Δp across a tornado is about 100 mb. Suppose $R = 1000$ m, which is not unreasonable for a tornado radius. What would the cyclostrophic wind speed be? You should find it is 100 m/s, or 224 mph. That would be a EF5 wind on the Enhanced Fujita scale we use to categorize tornadoes.