ATM 210 The Hydrostatic and Hypsometric Equations

Fall, 2023 – Fovell

Hydrostatic equation

Consider a molecule of air, as we did in lecture. There is a very strong **gravity force** attempting to bring the molecule down to the Earth's surface. This is being opposed by an upward-directed **pressure gradient force** (PGF). A pressure gradient is a pressure difference divided by a distance. PGF is always directed from high to low pressure. While pressure (and thus pressure gradients) can vary in any direction, the largest PGF is always pointing upwards, towards the zero pressure of outer space.

In **hydrostatic balance**, these two forces oppose and cancel. There is no net force, therefore no acceleration, up or down. *Hydro* comes from hydrogen and implies fluid. *Static* means stationary.

We can write the hydrostatic equation as:

$$\frac{dp}{dz} = -\rho g, \tag{1}$$

where p is pressure (in N/m^2 or Pa), z is height (m), ρ is density (kg/m^3), and g is the acceleration of gravity (about 9.81 m/s^2). Written in this fashion, each side resolves to units of N/m^3.

We can check the validity of the minus sign on the right hand side (RHS) in the following way. In the vertical direction, $\frac{dp}{dz}$ is always negative, because pressure decreases with height. In other words, p decreases in the $+z$ direction. As a consequence, the RHS has to be negative as well. But density is always positive, as is the acceleration of gravity, g. Thus, the minus sign is needed to make the equation work and be physically consistent.

By the way, the gravity acceleration g varies with z. That is a consequence of Newton's inverse square law. As you ascend in the vertical, you are moving away from the Earth's center of mass and, as a consequence, the gravitational attraction decreases a little bit. However, the atmosphere is not thick, and so the error associated with taking g to be a constant with z is not very large.

Aside: On Earth, g also varies with *latitude*. Think about why that might be.

Hypsometric equation

The **hypsometric** – *to measure height* – equation is obtainable using the hydrostatic and ideal gas equations:

$$z_2 - z_1 = \frac{R\bar{T}}{g} \ln \frac{p_1}{p_2}, \tag{2}$$

where z_1 and z_1 are two heights ($z_2 > z_1$), both in meters, R is the gas constant (J/kg/K), $\bar{T}$ is the mean temperature (K) of the layer between these heights, and p_1 and p_2 are pressures ($p_1 > p_2$) at the top and bottom. Since $p_1 > p_2$, the log of the ratio is positive. Thus, the RHS is positive and this confirms that $z_2 > z_1$.

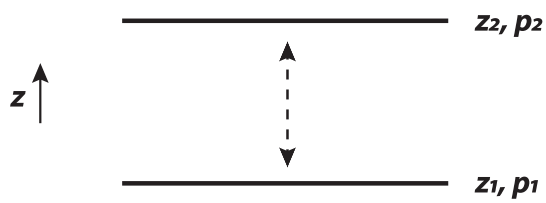

Figure 1: Thickness of a layer.

This equation is also called the **thickness equation** because the left-hand side (LHS) represents the thickness $\Delta z = z_2 - z_1$ between two levels or heights.

Suppose we're at the surface where $z_1 = 0$ and the pressure is $p_1 = 1000$ mb (= 100,000 Pa). We want to know the thickness of the 1000-500 mb layer and, therefore, roughly how far above the ground the $p_2 = 500$ mb = 50,000 Pa level resides. We need the layer mean temperature of that layer. Suppose the surface and 500 mb temperatures are 15 and -10°C, respectively. That's about 288 and 263 K. The arithmetic mean temperature is 275.5 K. So we have

$$\begin{aligned} z_2 - z_1 &= \frac{R\bar{T}}{g} \ln \frac{p_1}{p_2} \\ z_2 &= \frac{(287 \text{ J/kg/K})(275.5 \text{ K})}{9.81 \text{ m/s}^2} \ln \frac{100,000 \text{ Pa}}{50,000 \text{ Pa}} \\ z_2 &= (8060 \text{ m})(0.693) \\ z_2 &= 5587 \text{ m} \end{aligned}$$

The 500 mb level is about 5.6 km or 3.5 miles above the surface.

The dependence on temperature on the RHS illustrates an interesting and important point: *the thickness of a colder layer is less than that of a warmer layer.* Other factors being equal, decreasing T decreases z_2 relative to z_1.

We can use this equation, and our understanding, to help explain – directly or indirectly – such phenomena as why the troposphere is shallower in winter than in summer, shallower at the poles than at the equator, why the troposphere's coldest air is located at the tropical tropopause, and even why hurricane winds tend to be strongest very close to (although not right at) the ground. Another way of looking at that is: pressure decreases with height, but does so faster in colder air.

We can also use our understanding of the hypsometric equation to demonstrate my assertion that **temperature differences make pressure differences, and pressure differences drive**

winds. Consider two locations, A and B, having the same surface pressure $p_1 = 1000$ mb. Suppose the tropopause resides at the $p_2 = 200$ mb level above both locations. But the tropopause at location B resides farther above the surface. The same amount of mass is spread over a larger vertical depth. What is the difference between these two locations? The layer mean T above B is certainly higher, so $\Delta z = z_2 - z_1$ is larger.

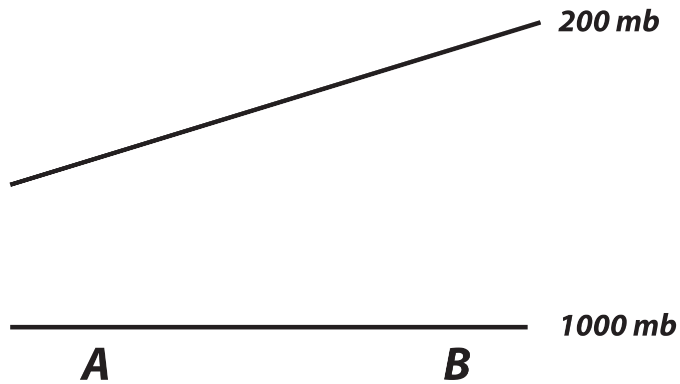

Figure 2: Same mass, greater thickness = higher mean temperature.

That's the temperature difference. Now look at the tropopause between the two locations. Do you see the pressure difference that results from this temperature difference? That pressure difference is trying to start a circulation – a wind.