1. Troy is 10 km due east from Albany airport. Let the point halfway between Albany and Troy be called “X”. The temperatures at Albany, X, and Troy are 10, 8, and 12°C, respectively.

(a) Compute the first derivative at point X. Express in degrees per kilometer.

\[
\begin{align*}
\text{Albany} & \quad 10^\circ C \\
X & \quad 8^\circ C \\
\text{Troy} & \quad 12^\circ C
\end{align*}
\]

\[
\begin{align*}
\Delta x & = 5 \text{ km, the distance between Albany and X, or X and Troy, so the distance between Albany and Troy is } 2\Delta x = 10 \text{ km.}
\end{align*}
\]

\[
f'(X) = \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2\Delta x} = \frac{12 - 10}{10} = 0.2 \text{ K/km.}
\]

(b) Compute the second derivative at point X. Express in degrees per square kilometer.

\[
f''(X) = \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}
\]

\[
= \frac{12 - 8 - 8 - 10}{5 - 5} = \frac{6}{25} \text{ K/km}^2
\]

(c) Briefly discuss why your calculations might contain error.

\textbf{Ans:} At the very least, both formulae are based on truncated Taylor series. The Mean Value Theorem states that they would be exactly correct (to within roundoff) somewhere in the interval, but we do not know where that point is. Truncation results in error if any of the neglected terms (representing higher order derivatives) are nonzero. In part (b), for example, you demonstrated that the second derivative neglected in part (a) is definitively nonzero.
2. The wind is from the west at 10 m/s. Temperature decreases to the north at 1\(^\circ\) per 100 km. As air flows, it is being heated by solar radiation at a rate of 1\(^\circ\)C/hr. Calculate the local time rate of change of temperature at a fixed point. Answer in K per hour.

(a) Draw a picture. Identify north. Sketch the wind as a vector, and draw two isotherms labeled T and T-\(\Delta T\).

\textbf{Ans:}

\begin{center}
\begin{tikzpicture}
  \draw[->] (0,0) -- (1,0) node[above] {T-\(\Delta T\)};
  \draw (0,0) -- (1,0) node[above] {T};
  \draw[->] (0,0) -- (0,0.5) node[above] {\(\nabla\)};
\end{tikzpicture}
\end{center}

Figure 2: Picture for \#1.

(b) The total derivative of temperature is

\[
\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{U} \cdot \nabla T,
\]

where \(\vec{U} = u\hat{i} + v\hat{j} + w\hat{k}\). Manipulate this equation to solve the problem at hand. What is the local time rate of change of temperature at the fixed point?

\textbf{Ans:} Hopefully, your picture makes it obvious that there is NO temperature advection. Therefore

\[
\frac{\partial T}{\partial t} = \frac{dT}{dt} = 1 \text{ K/hr}.
\]
3. A vector flow field is described by $\vec{U} = 3y\hat{i} - 3x\hat{j} + 0\hat{k}$. Compute the three-dimensional vorticity and divergence for this field. Sketch this flow in the x-y plane. Show your equations and steps clearly.

**Ans:** We have $u = 3y$, $v = -3x$ and $w = 0$. Divergence is

$$\nabla \cdot \vec{U} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

since $u$ is not a function of $x$, $v$ is not a function of $y$, and $w$ is zero.

For vorticity, we can expand out all three directions, or realize that two of them vanish because $w$ is zero and nothing varies with $z$. We are left with the vertical vorticity term

$$\nabla \times \vec{U} = \hat{k} \left[ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right] = \hat{k}(-3 - (3)) = -6\hat{k}.$$

The minus sign means we have anticyclonic (clockwise) vertical vorticity, so our sketch had better reveal clockwise rotation.

4. A baseball pitcher, standing about 18.3 m from home plate (60 ft. 6 in.), delivers a 42.5 m/s (95 mph) fastball to home plate in Wrigley Field. By how much is it deflected horizontally by the Coriolis force? The stadium is located at $\phi = 40^\circ$N latitude.

**Hint:** Your choice of horizontal direction is arbitrary. If the pitcher throws the ball eastward, then the Coriolis deflection is to the south, and the acceleration you want is

$$\frac{dv}{dt}_{\text{Corio}} = -2\Omega u_0 \sin \phi_0,$$

where $u_0$ is the baseball’s initial eastward speed and $\phi_0$ is the initial latitude. In physics, you learned about an equation of the form $x = x_0 + \frac{1}{2}at^2$, right?

**Ans:** We have $u_0 = 42.5$ m/s, and $f \approx 10^{-4}$ per second. The time taken from the mound to home plate is $t = 18.3/u_0 = 0.43$ s. Thus, the Coriolis acceleration, $fu_0 = 0.00425$ m/s$^2$.

Using $x = x_0 + \frac{1}{2}at^2$, and recognizing $x_0 = 0$, results in a deflection of 0.00039 m or 0.039 cm, to the right, of course.
5. The equations of horizontal motion involving only the Coriolis accelerations are

\[
\frac{du}{dt} = fv,
\]

and

\[
\frac{dv}{dt} = -fu.
\]

Suppose the Earth were a flat disk, rotating counterclockwise. In this case, \( f = 2\Omega \). Suppose further the rotation is \( |\vec{\Omega}| = 1 \) per hour. Shoot a rocket towards the outer edge at 10 km/h. The motions remain horizontal and there are no other forces acting than Coriolis. You measure a deflection of 100 km towards the right following the motion. How much time did this take? If you encounter a square root, you do not need to solve for it.

**Ans:** It actually does not matter which equation you select, but you don’t need more than one. I’ll select the first one, with the object initially traveling south \((v < 0)\). You can solve this one of two ways. First, you can integrate your selected equation twice, solving for position, which gets you to an equation such as that used in the question above. Or, as a shortcut, you can simply recognize that we’re approximating the acceleration as constant, and jump to that same equation:

\[
x = x_0 + \frac{1}{2}at^2.
\]

You want \( \Delta x = x - x_0 = -100 \) km; the initial position does not matter. \( \Delta x < 0 \) since the deflection is westward. \( a = \frac{du}{dt} = fu \), with \( v = -10 \) km/h (to the south) and \( f = 2|\vec{\Omega}| = 2/h \). So, \( a = -20 \) km/h\(^2\). Solve for \( t = \sqrt{10} \) h.