ATM 316 - The "thermal wind"

Fall, 2016 – Fovell

Recap and isobaric coordinates

We have seen that for the synoptic time and space scales, the three leading terms in the horizontal equations of motion are

$$\frac{du}{dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (1)$$

$$\frac{dv}{dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad (2)$$

where $f = 2\Omega \sin\phi$. The two largest terms are the Coriolis and pressure gradient forces (PGF) which combined represent geostrophic balance. We can define geostrophic winds u_g and v_g that exactly satisfy geostrophic balance, as

$$-fv_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (3)$$

$$fu_g \equiv -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad (4)$$

and thus we can also write

$$\frac{du}{dt} = f(v - v_g) \quad (5)$$

$$\frac{dv}{dt} = -f(u - u_g). \quad (6)$$

In other words, on the synoptic scale, *accelerations result from departures from geostrophic balance*.

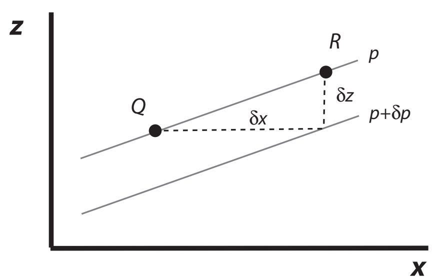

Figure 1: Isobaric coordinates.

It is convenient to shift into an *isobaric coordinate system*, replacing height z by pressure p. Remember that pressure gradients on constant height surfaces are height gradients on constant pressure surfaces (such as the 500 mb chart). In Fig. 1, the points Q and R reside on the same isobar p. Ignoring the y direction for simplicity, then $p = p(x, z)$ and the chain rule says

$$\delta p = \frac{\partial p}{\partial x}\delta x + \frac{\partial p}{\partial z}\delta z.$$

Here, since the points reside on the same isobar, $\delta p = 0$. Therefore, we can rearrange the remainder to find

$$\frac{\delta z}{\delta x} = -\frac{\frac{\partial p}{\partial x}}{\frac{\partial p}{\partial z}}.$$

Using the hydrostatic equation on the denominator of the RHS, and cleaning up the notation, we find

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -g\frac{\partial z}{\partial x}. \tag{7}$$

In other words, we have related the PGF (per unit mass) on constant height surfaces to a "height gradient force" (again per unit mass) on constant pressure surfaces. We will persist in calling this height gradient force a "PGF" or, more specifically, the isobaric PGF. Recalling geopotential $d\Phi = gdz$, we can also get

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{\partial \Phi}{\partial x}. \tag{8}$$

Similarly,

$$-\frac{1}{\rho}\frac{\partial p}{\partial y} = -\frac{\partial \Phi}{\partial y}. \tag{9}$$

Note density does not appear in the isobaric PGF. This is the principal advantage of isobaric coordinates.

Further, we can write an equation like (8) as

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\left(\frac{\partial \Phi}{\partial x}\right)_p \tag{10}$$

to remind ourselves that the height gradients are computed on isobaric surfaces. Therefore, the geostrophic wind equations on isobaric surfaces are

$$u_g = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial y}\right)_p \tag{11}$$

$$v_g = -\frac{1}{f}\left(\frac{\partial \Phi}{\partial x}\right)_p. \tag{12}$$

The hydrostatic equation in isobaric coordinates is

$$\frac{d\Phi}{dp} = -\frac{RT}{p}. \tag{13}$$

The thermal wind

Consider the familar situation shown in Fig. 2, in which the poles are colder than the equator. Suppose isobaric surface p tilts down towards the north. On a constant pressure surface, pressure value p resides closer to the surface, and represents a locale of lower geopotential height. Figure 3 depicts this situation. PGF points towards lower geopotential heights, Coriolis acts to the right following the motion (in the Northern Hemisphere), and thus the geostrophic wind blows parallel to isoheights with lower height to the left. It is a westerly wind in this example.

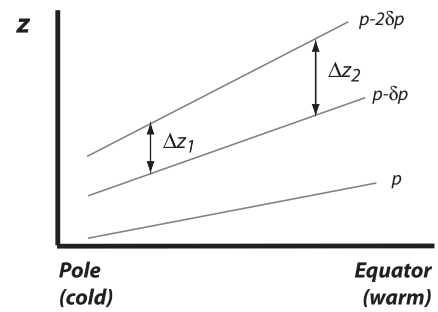

Figure 2: Pole colder than equator.

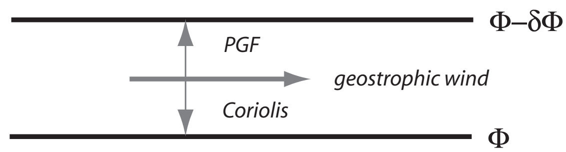

Figure 3: Geostrophic wind on isobaric surface p.

Also note thickness $\Delta z_1 < \Delta z_2$, since the average layer temperature of the former is lower. As a consequence, isobaric surfaces located farther aloft will slope progressively more strongly down towards the pole with height. This means the geopotential height gradient also increases, making the geostrophic wind stronger. (Note we don't have to consider density anymore; that's already implicitly factored in.) Thus, **not only does the geostrophic wind continue blowing from the west, there is also a westerly vertical wind shear**. Choosing a latitude residing somewhere in between pole and equator, this simple example gives us a vertical wind profile like this:

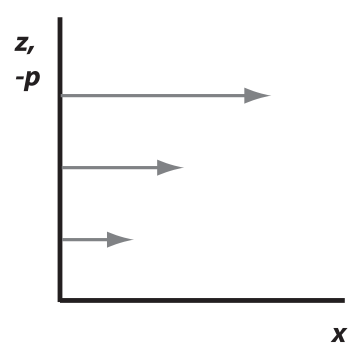

Figure 4: Westerly vertical shear of the geostrophic wind.

Therefore,*we can relate the vertical shear of the zonal (west-east) geostrophic wind to the meridional (north-south) temperature gradient*. Similarly, temperature gradients in the east-west direction imply vertical shear of the northerly component of the geostrophic wind. I am sticking with the north-south ∇T scenario merely because it's slightly easier to picture.

The "thermal wind" refers to the vertical shear of the geostrophic wind. In pressure coordinates, the vertical shear is

$$\frac{\partial u_g}{\partial p} \text{ and } \frac{\partial v_g}{\partial p}.$$

Since pressure decreases with height, $\frac{\partial u_g}{\partial p} < 0$ means u_g increases with height. Take (11) and differentiate it with respect to pressure. Thus

$$\frac{\partial u_g}{\partial p} = \frac{1}{f}\left[\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial y}\right)_p\right].$$

Interchange the order of differentiation on the RHS and use the isobaric hydrostatic equation (13) and find, after further rearrangement,

$$\frac{\partial u_g}{\partial \ln p} = \frac{R}{f}\left(\frac{\partial T}{\partial y}\right)_p. \tag{14}$$

That is, the vertical shear of the westerly geostrophic wind depends on the north-south ∇T. For the northerly geostrophic wind, we would find

$$\frac{\partial v_g}{\partial \ln p} = -\frac{R}{f}\left(\frac{\partial T}{\partial x}\right)_p. \tag{15}$$

If we define the vector geostrophic wind as $\vec{V}_g = u_g\hat{i} + v_g\hat{j}$, then

$$\frac{\partial \vec{V}_g}{\partial \ln p} = -\frac{R}{f}\hat{k} \times \nabla_p T, \tag{16}$$

where ∇_p reminds us to compute the temperature gradient on an isobaric surface. This is the thermal wind equation. It is NOT A WIND. It is a wind shear. Also, it does not involve the "true" wind, but rather the *geostrophic wind*. **If the wind is not geostrophic, then the thermal wind equation is not exact, and may even mislead**. Finally, it relates geostrophic shear to temperature gradients on isobaric surfaces – not constant height surfaces.

By the nature of the cross product, (15) also shows *the vertical shear is parallel to isotherms*, since it must be orthogonal to ∇T. We can see this more easily if we actually define a vertical shear vector $\vec{V}_T$ (where the subscript T stands for "thermal") by integrating between two isobaric surfaces p_0 and p_1

$$\vec{V}_T = \int_{p_0}^{p_1} \frac{\partial \vec{V}_g}{\partial \ln p} d\ln p = -\frac{R}{f}\int_{p_0}^{p_1} \hat{k} \times \nabla_p T,$$

The LHS is simply $\vec{V}_T = \vec{V}_g(p_1) - \vec{V}_g(p_0) \equiv u_T\hat{i} + v_T\hat{j}$, where u_T and v_T are the shear vector's components. The RHS is messy, but simplifies a lot if we take a layer mean T between the two pressure levels. So, the thermal wind shear component equations are

$$u_T = -\frac{R}{f}\left[\frac{\partial \bar{T}}{\partial y}\right]_p \ln\frac{p_0}{p_1} \tag{17}$$

$$v_T = \frac{R}{f}\left[\frac{\partial \bar{T}}{\partial x}\right]_p \ln\frac{p_0}{p_1}. \tag{18}$$

The hypsometric equation permits us to rewrite the RHS of the above as

$$u_T = -\frac{1}{f}\frac{\partial}{\partial y}\left[\Phi_1 - \Phi_0\right] \quad (19)$$

$$v_T = \frac{1}{f}\frac{\partial}{\partial x}\left[\Phi_1 - \Phi_0\right]. \quad (20)$$

There are three elements in the preceding. For the p_0 to p_1 layer, there is the geostrophic wind at the layer bottom $\vec{V}_g(p_0)$, the geostrophic wind at the layer top $\vec{V}_g(p_1)$, and the vertical shear $\vec{V}_T$, itself determined by the horizontal gradient of layer mean temperature. If we know any two of these, we can get the third. Keep in mind that since u_T is proportional to $\frac{\partial \bar{T}}{\partial y}$ and v_T is proportional to $\frac{\partial \bar{T}}{\partial x}$ that $\vec{V}_T$ is parallel to isotherms in the p_0 to p_1 layer.