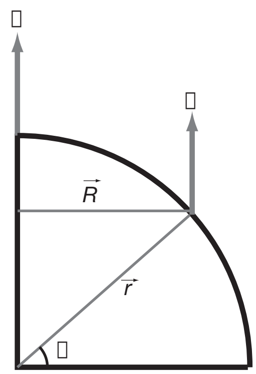
ATM 316 - Why is $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ centripetal acceleration?

Fall, 2016 – Fovell

This is a proof of the assertion that $\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{R}$, which is the centripetal acceleration.

Refer to the picture below, which shows the Earth's rotation vector $\vec{\Omega}$ positioned at a midlatitude location. Also shown are $\vec{r}$ and $\vec{R}$, vectors to that position from the Earth's center and spin axis, respectively. First, keep in mind that when you use the cross product, you only deal with *orthogonal* components. Note that the component of $\vec{r}$ perpendicular to $\vec{\Omega}$ is $\vec{R}$! So immediately the problem becomes resolving $\vec{\Omega} \times (\vec{\Omega} \times \vec{R})$.



Next, we make use of the vector identity:

$$\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3) = (\vec{V}_1 \cdot \vec{V}_3)\vec{V}_2 - (\vec{V}_1 \cdot \vec{V}_2)\vec{V}_3,$$

resulting in

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{R}) = (\vec{\Omega} \cdot \vec{R})\vec{\Omega} - (\vec{\Omega} \cdot \vec{\Omega})\vec{R}.$$

Since $\vec{\Omega}$ is perpendicular to $\vec{R}$, $\vec{\Omega} \cdot \vec{R} = 0$! Since $\vec{\Omega} \cdot \vec{\Omega} = \Omega^2$, we have established that

$$\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) = -\Omega^2 \vec{R}.$$

directed inward, towards the center of spin. This is the centripetal acceleration.