ATM 562 (Fovell) – Fall, 2021

Homework solution

Someone proposes to use a forward-time, center-space approximation to the 1D constant advection equation

$$u_t + cu_x = 0$$

given by

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\left[\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}\right] = 0.$$

1. Is this scheme consistent with the original PDE? (It will suffice to note whether or not the truncation error goes to 0 as Δt and Δx go to zero, without the complicated substitutions for the time derivatives we employed in the upstream scheme demonstration.)

   Solution: Yes, it is consistent. We use these Taylor expansions about U_j^n:

$$U_j^{n+1} = U_j^n + \Delta t u_t + \frac{\Delta t^2}{2!}u_{tt} + \cdots$$
$$U_{j+1} = U_j^n + \Delta x u_x + \frac{\Delta x^2}{2!}u_{xx} + \frac{\Delta x^3}{3!}u_{xxx} + \cdots$$
$$U_{j-1} = U_j^n - \Delta x u_x + \frac{\Delta x^2}{2!}u_{xx} - \frac{\Delta x^3}{3!}u_{xxx} + \cdots.$$

   Substituting these into our scheme, we obtain after some simplification:

$$u_t + cu_x = -\frac{\Delta t}{2}u_{tt} - c\frac{\Delta x^2}{3!}u_{xxx} + \cdots.$$

   The left hand side is our differential equation, and the right hand side represents our truncation error, which is first order in time and second order in space. As demonstrated with the upstream scheme, we *should* work on converting the time derivatives in the truncation error into space derivatives, so we can see that the right hand side goes to zero with Δt and Δx as $\frac{\Delta t}{\Delta x}$ remains constant. However, that said, it should be readily apparent that the right hand side *will* disappear as the time step and grid spacing is decreased, so consistency is demonstrated.

2. Whether or not the scheme is consistent, determine the stability condition for this scheme, using the modified Von Neumann approach discussed in class and in the course notes.

   Solution:

   In explicit form, this scheme is

$$U_j^{n+1} = U_j^n - \frac{c\Delta t}{2\Delta x}\left[U_{j+1}^n - U_{j-1}^n\right].$$

   Define as usual $U = Ae^{-i\omega t}$, $\lambda = e^{-i\omega\Delta t}$, $U_j^{n+1} = U_j^n\lambda$, $U_{j+1}^n = U_j^n e^{ik\Delta x}$ and $U_{j-1}^n = U_j^n e^{-ik\Delta x}$. Also, for this situation, let $c' = \frac{c\Delta t}{2\Delta x}$. Therefore,

$$\begin{aligned} \lambda &= 1 - c'\left[e^{ik\Delta x} - e^{-ik\Delta x}\right] && (1) \\ \lambda &= 1 - c'\left[2i\sin k\Delta x\right] && (2) \end{aligned}$$

   in which the real part is 1 and the imaginary part is $c'\left[2\sin k\Delta x\right]$. So squaring the real and imaginary parts separately, we find

$$|\lambda|^2 = 1 + \left[2c'\sin k\Delta x\right]^2$$

   Since the term being added to 1 on the right hand side is squared, and thus never negative, it is clear that $|\lambda| > 1$ for all nonzero (i.e., reasonable) values of c'. Thus, the scheme is always absolutely unstable.