ATM 562 (Fovell) – Fall, 2021

Homework #2 Solutions

1. Show that this scheme is linearly unstable:

$$\frac{U^{n+1} - U^{n-1}}{2\Delta t} = -aU^n.$$

Solution:

In explicit form, this scheme is

$$U^{n+1} = U^{n-1} - 2a\Delta t U^n.$$

Define as usual $U = Ae^{-i\omega t}$, $\lambda = e^{-i\omega\Delta t}$, $U^{n+1} = U^n\lambda$, and $U^{n-1} = U^n/\lambda$. Therefore,

$$\begin{aligned} \lambda &= 1/\lambda - 2a\Delta t && (1) \\ \lambda^2 + 2a\Delta t\lambda - 1 &= 0 && (2) \\ \lambda &= -a\Delta t \pm \sqrt{a^2\Delta t^2 + 1}. && (3) \end{aligned}$$

Let $x = a\Delta t$. Then, $\lambda = -x \pm \sqrt{x^2 + 1}$. All we need to demonstrate is that the scheme is unstable is if one of the two roots is always > 1. This will be true for the negative root:

$$\begin{aligned} \lambda_- &= -x - \sqrt{x^2 + 1} && (4) \\ |\lambda_-|^2 &= 1 + 2x^2 + 2x\sqrt{x^2 + 1}. && (5) \end{aligned}$$

For $x > 0$, which is necessary for a damper, all three terms on the right hand side are > 0, and one of the terms is 1 itself, so

$$|\lambda_-| > 1$$

always. The scheme is absolutely unstable.

2. Two $3\Delta x$ waves interact nonlinearly. What *unresolvable* wavelength does this interaction create? What is the wavelength of its aliased counterpart?

   Solution:

   Let $k_1 = k_2 = \frac{2\pi}{3\Delta x}$. Through nonlinear interaction, the sum of these wavenumbers yields a smaller wavelength than represented by the originals. Thus, we have $\hat{k} = k_1 + k_2 = \frac{4\pi}{3\Delta x} = \frac{2\pi}{3/2\Delta x}$, or a $1.5\Delta x$ wave that is unresolvable. This wave aliases to

$$k^* = 2k_{max} - \hat{k} = \frac{2\pi}{3\Delta x}.$$

   Thus, two $3\Delta x$ waves can combine to retain energy at its original wavelength.