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Frequency Filter for Time Integrations

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ABSTRACT—A simple filter for controlling high-frequency computational and physical modes arising in time integrations is proposed. A linear analysis of the filter with leapfrog, implicit, and semi-implicit differences is made. The filter very quickly removes the computational mode and is also very useful in damping high-frequency physical waves. The stability of the leapfrog scheme is adversely affected when a large filter parameter is used, but the analysis shows that the use of centered differences with frequency filter is still more advantageous than the use of the Euler-backward method. An example of the use of the filter in an actual forecast with the meteorological equations is shown.

1. INTRODUCTION

Certain types of finite-difference approximations for time derivatives have traditionally been used for integrating hydrodynamical equations because of their known damping properties (Kurihara 1965). Often, as in the Euler-backward method, the amount of damping is not adjustable and the calculations involve extra evaluations of the tendency, making the procedure very costly. At the Dynamic Prediction Research Unit we have been using a frequency filter originally designed by Robert (1966) that alleviates both of these inconveniences. An analysis of this filter for a simple advective equation treated with centered implicit or explicit differences is presented below, and response curves for several values of the filter parameter are shown. Particular attention is given to the study of the linear stability of the leapfrog scheme. Examples of various uses of the filter are given in section 4.

2. BASIC TIME FILTER

Assume a time series

$$F(t)=e^{i\omega t} \tag{1}$$

to be defined at discrete points one time unit apart with angular frequency, ω. The filtered function is

$$\overline{F(t)}=F(t)+0.5\nu[\overline{F(t-1)}-2F(t)+F(t+1)] \tag{2}$$

where ν is the filter parameter. Letting

$$\overline{F(t)}=\overline{X}F(t) \tag{3}$$

define the response, $\overline{X}$, of the filter, we find that

$$\overline{X}=\frac{(2-\nu)^2+2\nu^2(1-\cos\omega)e^{i\omega}}{(2-\nu)^2+4\nu(1-\cos\omega)}=R(\nu,\omega)e^{i\delta(\nu,\omega)}. \tag{4}$$

The amplitude response, R, and the phase shift, δ, are plotted against $\cos\omega$ for different values of ν in figures 1 and 2. One can see that the responses of all filters with parameters between 0.5 and 1 are very similar, some kind of optimum being achieved with $\nu\approx 6/7$. In general, for moderate values of ν, the phase shift is large only for highly damped waves.

We see also that for $0<\nu<2/3$ the amplitude response, R, is similar to the response, R_c, of the well-known centered filter

$$\overline{F_c(t)}=F(t)+0.5\nu[F(t-1)-2F(t)+F(t+1)] \tag{5}$$

where

$$R_c=1-\mu(1-\cos\omega). \tag{6}$$

3. FILTER WITH TIME DIFFERENCES

The effect of the filter used in conjunction with leapfrog, semi-implicit, or fully implicit time differences will now be studied. Following Kurihara (1965), we decompose the frequency into two arbitrary parts and write the differential equation,

$$\frac{\partial F}{\partial t}=i\omega F, \tag{7}$$

in difference form as

$$\frac{F(t+1)-\overline{F(t-1)}}{2}=i\omega_A F(t)+i(\omega-\omega_A)\frac{[F(t+1)+\overline{F(t-1)}]}{2}. \tag{8}$$

In eq (8), the smoothed function at $t-1$ is obtained from eq (2) using values at $t-2$, $t-1$, and t. If we now define the amplification factor by

$$F(t+1)=XF(t), \tag{9}$$

we find from eq (8) and (2) that

$$X_{1,2}=\frac{\nu+2i\omega_A\pm[(\nu-2)^2+4\omega^2(1-\nu)+4\omega\omega_A(\nu-2)]^{0.5}}{2(1-i\omega+i\omega_A)}. \tag{10}$$

The negative sign corresponds to the computational solution.

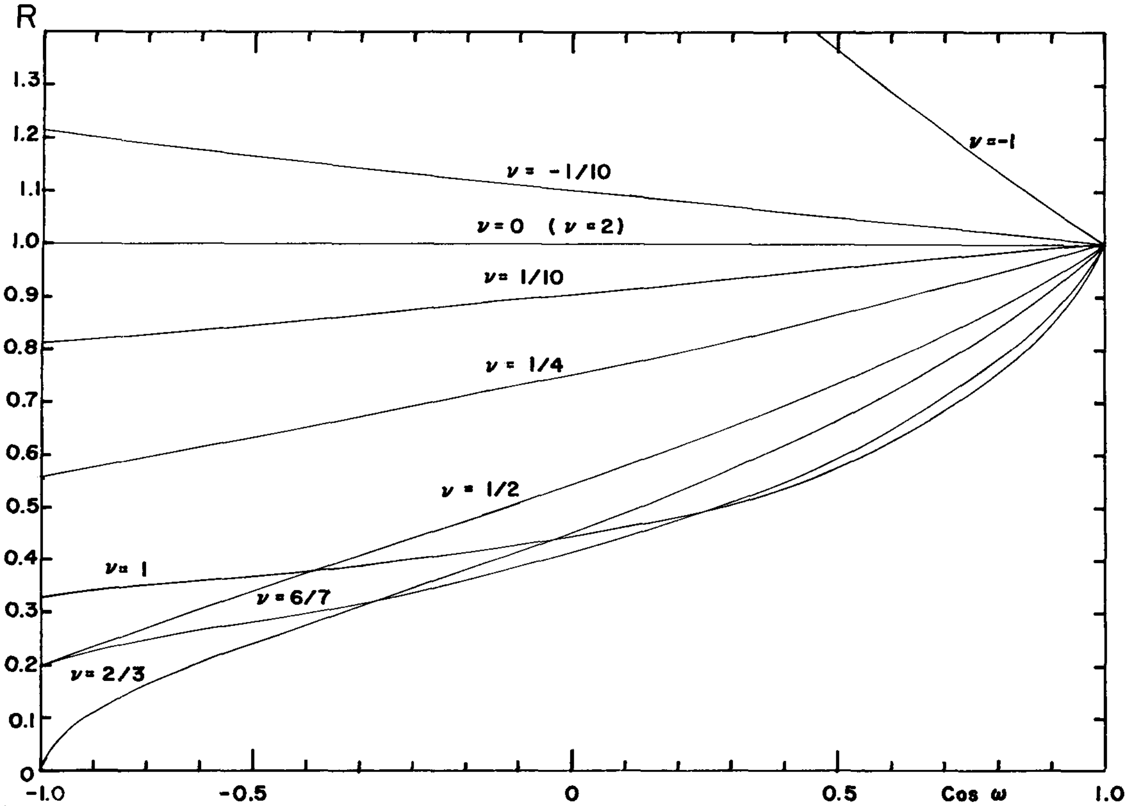

FIGURE 1.—Amplitude response due to application of the time filter to a periodic function for a few values of the filter coefficient. The response of a conventional centered filter is a straight line with intercepts $1-\nu$ at $\cos\omega=0$ and 1 at $\cos\omega=1$.

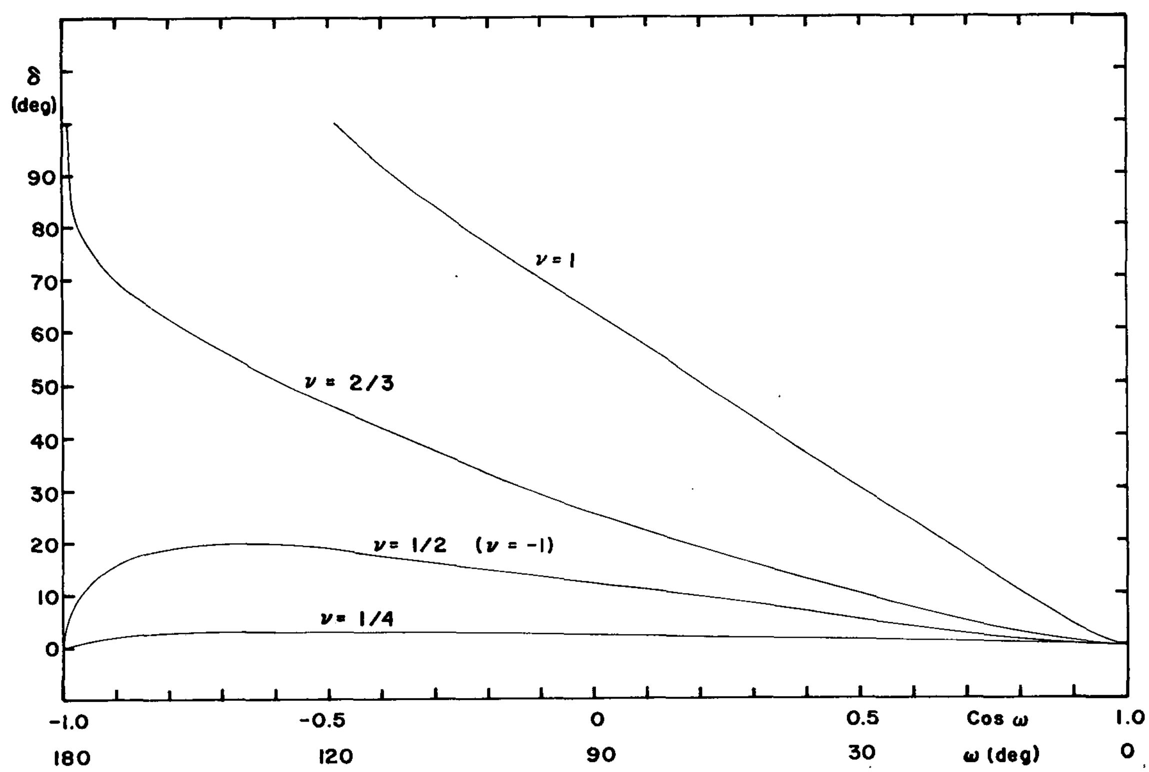

FIGURE 2.—Phase increase due to application of the time filter to a periodic function for a few values of the filter coefficient.

By setting $\omega_A=0$ in eq (10), we get the amplification factor applicable to the implicit scheme; that is,

$$X=\frac{1+i\omega}{1+\omega^2}\left[\frac{\nu}{2}\pm\sqrt{1+\omega^2+\frac{\nu^2}{4}-\nu(1+\omega^2)}\right]. \qquad (11)$$

For $\nu\leq 1$, the radicand is positive so that the frequency is $\tan^{-1}\omega\Delta t$ independently of ν. Also, by using $(\sqrt{1+\omega^2}-\nu/2)^2$ as an upper bound for the radicand we see that $|X|<1$ for $\nu>0$ and $\omega>0$.

If, instead, we set $\omega=\omega_A$, we get the leapfrog scheme

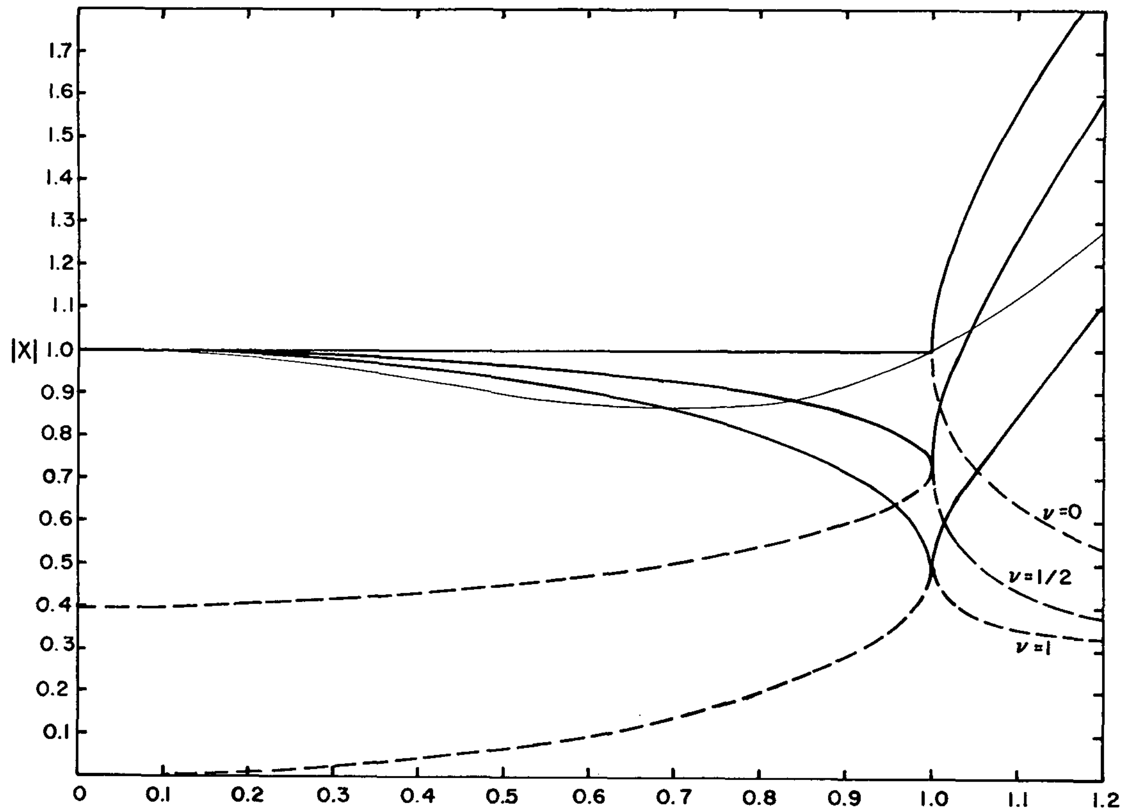

FIGURE 3.—Size of the amplification factor against frequency for the Euler-backward method (thin line) and for the leapfrog scheme including filter with coefficients $\nu=0$, 0.5, and 1.0 with $\Delta t=1.0$, 0.75, and 0.5, respectively. The dashed parts belong to the computational solution.

for which

$$X=\frac{\nu}{2}+i\omega\pm\sqrt{\left(1-\frac{\nu}{2}\right)^2-\omega^2}. \tag{12}$$

When $\omega^2\leq(1-\nu/2)^2$, the largest root has magnitude smaller than 1. The stable range is slightly greater, however, and by allowing the radicand to become negative we find stability ($|X|\leq 1$) for

$$\omega^2+\omega\sqrt{\omega^2-\left(1-\frac{\nu}{2}\right)^2}-\left(1-\frac{\nu}{2}\right)<0. \tag{13}$$

Thus, with $\mu=1$, we need $\omega<0.57$; that is, we must reduce the time step, Δt, to 0.57 time unit if we want to admit the same range of frequencies as the leapfrog scheme alone in the stable domain of X.

It is interesting to compare the characteristics of the leapfrog scheme including a heavy filter with another well-known dissipative scheme, the Euler-backward method for example. The amplification factor for the latter is

$$A_{EB}=1-\omega^2+i\omega;$$

with $\Delta t=1$ this scheme is stable for $\omega\leq 1$. For the leapfrog scheme with filter, we select $\nu=1$ and $\Delta t=0.5$ for simplicity (although the maximum time step would be 0.57). The amplification per unit time is then

$$A_{\nu=1}=\frac{(1\pm\sqrt{1-\omega^2}+i\omega)^2}{4}$$

since there are two time steps per unit time. The amplitudes of A_{EB} and $A_{\nu=1}$ are plotted against ω in figure 3. There are also two other curves in figure 3:

$$A_{\nu=\frac{1}{2}}=\left[\frac{(1\pm 3\sqrt{1-\omega^2}+3i\omega)}{4}\right]^{4/3}$$

and

$$A_{\nu=0}=\pm\sqrt{1-\omega^2}+i\omega$$

corresponding to two other choices of ν and Δt ($\nu=\frac{1}{2}$, $\Delta t=\frac{3}{4}$ and $\nu=0$, $\Delta t=1$). From these, the curve for any other choice of ν can easily be imagined.

For any value of $\nu\leq 1$, the filter with leapfrog differences gives less damping of the low frequencies than the Euler-backward method. On the other hand, depending on the choice of ν, more damping can be obtained at high frequencies. The damping per time step is actually very nearly equal to the damping of the frequency filter [eq (2)] alone. As for the computational solution, it is nicely damped for any choice of $\nu>0$. Since the Euler-backward method requires the evaluation of two tendencies per unit time, the use of the filter with the leapfrog scheme remains advantageous.

When the time filter is used with centered differences, the computed frequency is increased by approximately $\delta/\Delta t$, where δ is the phase shift of the filter alone as in eq (4). However, this effect is almost perfectly offset if the time step is reduced by $1-\nu/2$. In particular, if $\nu=1$ and $\Delta t=0.5$ as in the example above, the frequency is identical

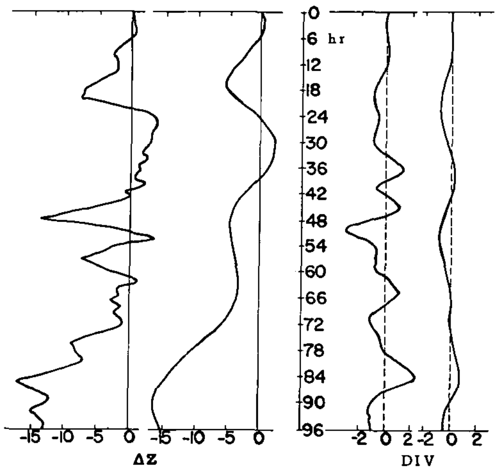

FIGURE 4.—The 500-mb height change (m) and divergence ($10^{-6}s^{-1}$) at a point near the border during the integration of a three-level, primitive-equation model without filter (rough curves) and with a time filter using coefficients $\nu=0.86$ up to 36 hr, $\nu=0.5$ to 48 hr, $\nu=0.2$ to 60 hr, and $\nu=0$ thereafter (smooth curves).

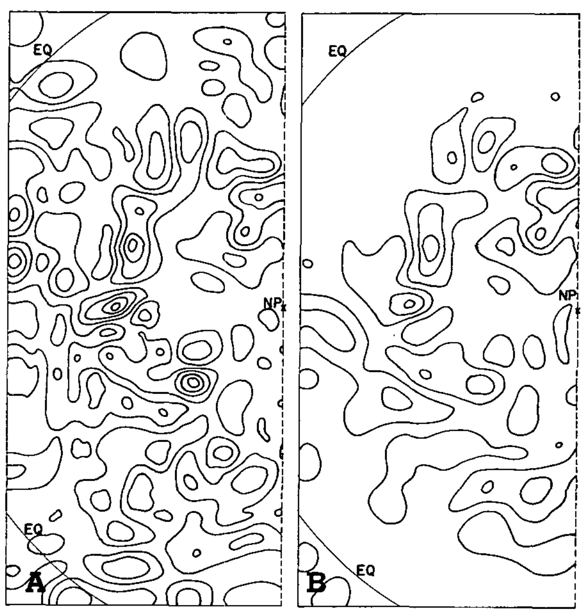

FIGURE 5.—The 500-mb divergence after 96 hr of integration (A) without filter and (B) with the time filter.

to that of the leapfrog scheme alone ($\nu=0$, $\Delta t=1$) since

$$\tan^{-1}\frac{\omega}{\sqrt{1-\omega^2}}=2\tan^{-1}\frac{\omega}{1+\sqrt{1-\omega^2}}, \qquad \text{for } \omega<1.$$

To complete the study of the frequency filter, we calculated the amplification factors from eq (10) for three values of ω_A/ω (0, 0.2, 1.0) and for several values of ν from -1.0 to 5.0. These calculations supported the above analysis and showed that the range of ν should be limited to the interval (0, 1).

4. EXAMPLES

The frequency filter has been used for various purposes so far. Robert (1966) used it first in a general circulation primitive-equation spectral model with centered differences to control the instability due to a friction term; he was using $\nu=0.02$ and $\Delta t=20$ min. Kwizak (1970) prevented the growth of the computational solution in a 20-day semi-implicit integration by using $\nu=0.01$ and $\Delta t=60$ min.

One of the interesting applications of the filter has been for the elimination of a significant part of the spectrum of external and internal gravity waves in a hemispheric primitive-equation baroclinic model. A large parameter was used to quickly damp the oscillations shorter than about 12 hr without affecting significantly those longer than 24 hr. The parameter was 0.86 for the first 36 hr, 0.5 until 48 hr, 0.2 until 60 hr, and finally 0.0 thereafter. The time steps were 1 hr, the grid length 381 km at 60°N on a polar stereographic projection, and the integration was by the semi-implicit method. The model was a baroclinic extension of the model described by Kwizak and Robert (1971). Figure 4 shows the time variation of the geopotential and of the divergence at 500 mb at one gridpoint during an integration (1) when the filter parameter was zero and (2) when the time filter with the parameters given above was applied. Figures 5A and 5B show the left half of the 96-hr forecast of the divergence in these two cases.

5. CONCLUSIONS

The frequency filter described earlier is an excellent damper for the computational modes arising in leapfrog, centered implicit, and centered semi-implicit time integrations. Furthermore, because it discriminates well between frequencies, it can be adjusted to damp out high-frequency motions in the physical solution such as external gravity-inertia waves. The possibility of varying the filter parameter during an integration makes it a very versatile tool. The application of the filter requires very little computing time, although it does require the storage of the time-dependent variables at three time levels. The filter does not badly affect the stability of the difference equations for reasonable values of the filter coefficient.

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