Group Velocity and the Linear Response of Stratified Fluids to Internal Heat or Mass Sources

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ABSTRACT

A steadily maintained line heat or mass source turned on in an unbounded, steadily moving, uniformly stratified flow will in general create ever-increasing vertical displacements of the fluid. Lin and Smith viewed a maintained heat source as a train of heat pulses. A pulse occurring a time T before the observation time creates a negative displacement proportional to T^{-1} at the heat source position when T is large. They pointed out that superposing the pulse responses leads to a displacement that grows logarithmically with time.

This paper uses group velocity arguments to recreate the gravity wave field a time T after a heat pulse. The T^{-1} decay of the displacement is shown to be a geometrical consequence of dispersion in two dimensions. The growing response to a maintained source can be understood as the result of energy being pumped into the gravity wave modes, whose group velocity is near zero, faster than it can spread in physical space due to dispersion. A steady response is shown to be possible only if the heat source distribution has no projection onto the modes of zero group velocity. If the fluid is bounded both above and below, the vertical wavenumbers of gravity wave modes are quantized. Unless the layer depth is resonantly tuned, there are no normal modes of zero group velocity and a steady response develops.

The same arguments allow the work of Smith and Lin to be generalized to more complicated situations, e.g., when there is either ambient rotation or localization of the heat source in all three dimensions, and show that a steady state will develop in response to a maintained heat source in these cases because the response to a pulse heat source decays faster than T^{-1}. Analogous results hold for a mass source or flow over a ramp. Only very large vertical displacements or wave breaking are likely to alter these conclusions.

1. Introduction

A number of meteorological phenomena can be modeled as the response of a stably stratified atmosphere moving with respect to a heat or mass source. Examples involving heat sources include orographically forced precipitating clouds (Fraser, et al., 1973; Barcilon et al., 1980; Smith and Lin, 1982), flow over heated islands (Garstang et al., 1975) and urban heat islands, and the modification by squall lines and mesoscale convective systems of their environment (Raymond, 1986). The response to a vertically specified heat source is also basic to theories of wave-CISK (Raymond, 1983). Flow up onto a plateau or over the nose of a steady density current can be idealized by replacing the windward slope of the obstacle by an equivalent mass source.

One can idealize several of these situations as localized fixed line heat sources placed in an unbounded steadily moving, stably stratified fluid. We use the following terminology: A "line" heat (or mass) source is one which is uniform in the horizontal direction y, that is, transverse to the direction x of the mean flow. It is "fixed" if it does not move with the fluid, but always heats in the same place with respect to some other reference frame, which we will call the "fixed" frame. The frame moving with the mean fluid velocity is called the "advected" frame. For orographically induced heating, the "fixed" frame does not move with respect to the earth's surface. However, for a density current moving into a stagnant air mass, the "fixed" frame would move with the nose of the density current, and the advected frame would be stationary with respect to the earth's surface. The heat source can be either "localized", if all of the heating takes place at one point in x and z or "distributed" otherwise. We will consider heat "pulses" in which all the heating occurs at one time and "maintained" heat sources, which are turned on at some time $t = 0$ and are steady thereafter.

Theoretical arguments (Smith and Lin, 1982; Lin and Smith, 1986, hereafter referred to as LS) show that a maintained line heat source in a moving unbounded Boussinesq fluid forces vertical displacements which do not become steady, but increase logarithmically with time. Lin and Smith found that the energy for the disturbance is produced by a downward displacement of fluid as it passes through the region of heating. This reinforces the disturbance by heating fluid that is al-

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ready warm. A similar unbounded transient response occurs as a stratified fluid flows over a ramp (Klemp, personal communication). There is a direct analogy between two-dimensional stratified flow and two-dimensional unstratified rotating flow (see, for instance, Yih, 1965, chap. 6). Applied to Lin and Smith's result, this analogy implies that in a rotating fluid moving uniformly in a direction perpendicular to the rotation axis, a fixed linear mass source aligned perpendicular to both the rotation axis and the current direction will also produce a growing response due to the buildup of energy in inertial waves.

This response is rather surprising. If a heat source is placed in a stratified fluid with buoyancy frequency N flowing at a speed U between rigid plates a height H apart, then the transient response remains finite unless NH/U is an integer multiple of π, when a resonance occurs. One might expect that the vertical radiation of energy by gravity waves forced by the heat source would if anything diminish the response.

In this article, we will use group velocity arguments based on the dispersion relation for gravity waves to understand the response to pulse and maintained heat and mass sources in a moving stratified Boussinesq fluid. In section 2a and 2b, we recapitulate the arguments of LS, deriving from their results a formula for the displacement due to a localized line heat pulse and showing how a maintained heat source can be regarded as a succession of small heat pulses whose effects are superposed. In section 2c, we show how the response to a heat pulse can be completely understood using group velocity arguments based on the dispersion relation for gravity waves. We will show that it is the continuous forcing of wavenumbers near $\pm\mathbf{k}_0 = (0, \pm N/U)$ by a maintained line heat source which is crucial to the ever-growing transient response in an unbounded domain or a domain only bounded below. These wavenumbers have very small group velocities and cannot rapidly disperse away the energy that is fed into them. The technique allows SL's line of argument to be generalized to check whether any localized source of energy in a medium permitting dispersive waves will lead to a steady state response. The flavor of the arguments is similar to those of Bretherton (1967), who showed how the linear response to a cylinder that suddenly starts moving transverse to the rotation axis of a rotating fluid can be thought of as a superposition of inertial waves that gradually act to produce a Taylor column along the rotation axis.

In section 3, we discuss the effects of distributing the source over a large area in x and z. We argue that the heating distribution will produce a steady response only if it has no projection on the wavenumbers $\pm\mathbf{k}_0$ associated with waves of zero group velocity. We discuss the effect of boundaries and of vertically varying N and U. The primary effect of having two boundaries is to discretize the allowed vertical wavenumbers. Unless $NH/U = n\pi$, there are no waves of zero horizontal group velocity in which energy can pile up, and a maintained line source will force a steady response.

Section 4 extends the arguments to a mass source. In section 5, we use the method of section 2 to show that a maintained heat source of finite extent in the transverse direction does allow a steady state to set up. This problem, unlike the line heat source problem, does not appear to have closed form solutions for particular heating profiles, so that the group velocity arguments provide a source of information not duplicated by other means. Similarly, a line heat source in a moving fluid on a rotating earth is also shown to lead to a steady state and the amplitude of the vertical displacement is estimated. The infinite response appears to be rather special to two-dimensional flow in a nonrotating atmosphere. Section 6 presents the conclusions.

2. The hydrostatic response to a localized pulse of heat

a. Basic equations

We will follow the tradition of most theoretical studies of gravity waves use the Boussinesq approximation (Spiegel and Veronis, 1961), and assume that the fluid density can be regarded as a constant ρ_0 except that it leads to a vertical acceleration, the buoyancy $B = -g\rho'/\rho_0$, where ρ' is the density perturbation from some chosen reference state depending only on z. Most atmospheric motions are more accurately described by the anelastic equations (Ogura and Phillips, 1962). In hindsight, we will see that our results carry over directly to a compressible atmosphere with mean density $\rho(z)$ because the inverse density scale height ($\sim 10^{-4}$ m^{-1}) is usually small compared to the vertical wavenumber N/U ($\sim 10^{-3}$ m^{-1} for $N = 10^{-2}$ s^{-1} and $U = 10$ m s^{-1}) of the waves which are most important in determining the response to a maintained heat source, so that these waves have almost the same dispersion relation as for a Boussinesq fluid with the same buoyancy frequency. One need only multiply all perturbation velocities and displacements by $[\rho(z)/\rho_0]^{-1/2}$. In the same spirit, we will make the hydrostatic approximation because it somewhat simplifies the gravity wave dispersion relation. We will again see that all of our results apply just as well even if the hydrostatic approximation were not made, because the waves which dominate the response to a maintained heat source have small horizontal wavenumbers $k \ll N/U$ and are very well described by the hydrostatic approximation.

Consider a fluid with constant buoyancy frequency N moving at a constant speed U in the x direction. We phrase the heating in terms of a buoyancy source because the buoyancy is the dynamically relevant quantity. The Boussinesq hydrostatic momentum, continuity and buoyancy equations for the perturbation velocities u, w, the perturbation pressure p, and the perturbation buoyancy B about this state due to a buoyancy source $q(x, z, t)$ are:

$$u_t + Uu_x = -p_x/\rho_0, \quad (1a)$$

$$0 = B - p_z/\rho_0, \quad (1b)$$

$$u_x + w_z = 0, \quad (1c)$$

$$B_t + UB_x + N^2 w = q. \quad (1d)$$

b. The response to localized and distributed line buoyancy pulses and maintained sources

Consider the response to a localized buoyancy pulse

$$q(x, z, t) = q_0\delta(x)\delta(z)\delta(t); \quad (2)$$

δ is the Dirac delta function. An extension of this problem was considered by SL, who considered a pulse source of heating at the single height $z = 0$, but distributed in the horizontal with a half-width b. Mathematically, they chose $q(x, z, t) = (C_pT_0/g)(Q_0b^2/[x^2 + b^2])\delta(z)\delta(t)$, where C_p is the isobaric heat capacity of air, T_0 is a reference temperature, and Q_0 is a heating rate. The factor C_pT_0/g, which does not appear in their definition of the heating rate, rescales a heating rate into a rate of buoyancy production. The total amount of buoyancy added by the heating integrated over all x and z, the "strength" of the pulse, is $q_0 = (C_pT_0/g)\pi Q_0 b$. Interpreted as a distribution, $b/(x^2 + b^2) \to \pi\delta(x)$ as $b \to 0$. Hence one may obtain the vertical displacement $\eta_{1p}(x, z, t)$ in response to a localized buoyancy pulse by replacing $(C_pT_0/g)\pi Q_0 b$ by q_0 and taking the limit of their Eq. (7) as $b \to 0$ and $Q_0 \to \infty$ such that q_0 remains fixed:

$$\eta_{1p}(x, z, t) = -(q_0t/2\pi NX^2)\cos(Nzt/X), \quad (3)$$

where $X = x - Ut$ is the horizontal coordinate in the advected frame. Viewed in the advected frame, (3) is just the response to a heat pulse in an initially stationary fluid. We will stick to the fixed frame because of our focus on fixed heat sources. A snapshot of the displacement field at a particular time is shown in Fig. 1. In section 2c we will show how this figure can be fully reconstructed from the gravity wave dispersion relation. At the heating level $z = 0$, $\eta_{1p}(x, 0, t) = -q_0t/2\pi NX^2$. Note that (3) breaks down at $X = z = 0$. To compensate the downward displacement at other x's, there must be concentrated rising motion occurring at $x = Ut$ ($X = 0$) in the heated fluid as it is advected along by the mean flow. In anticipation of superposing the pulse response from various times to obtain the response to a fixed maintained buoyancy source, it is particularly important to find the displacement at the source position S ($x = z = 0$):

$$\eta_{1pS}(t) = \eta_{1p}(0, 0, t) = -q_0/2\pi NU^2t. \quad (4)$$

Consider a distributed heat pulse $q_d(x, z)\delta(t)$ of the same strength q_0 but finite characteristic width Δx and height Δz. The strength is the integral of $q_d(x, z)$ over all space. One can regard this source as a spatial integral of localized pulses. The response at S is almost the same as for a single localized pulse when the responses

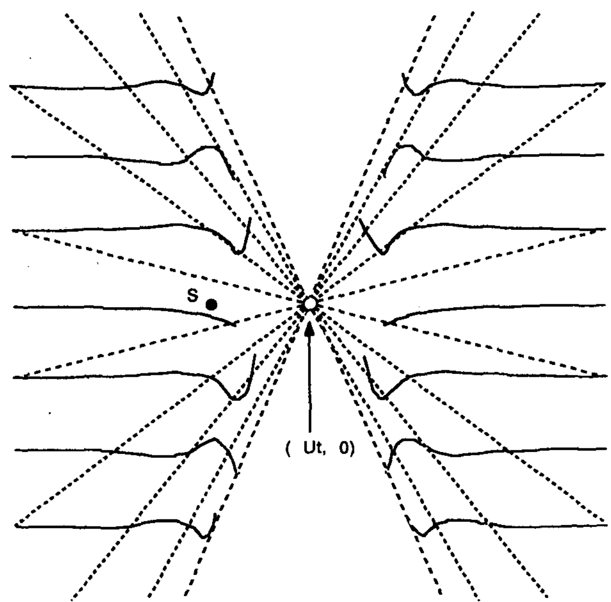

FIG. 1. The displacement field at time t after a localized impulsive buoyancy source in a stratified fluid. The dashed lines are the nodal lines $z = (n + 1/2)\pi X/Nt$ on which there is no vertical displacement. The solid lines are dye lines that were initially horizontal and have been displaced in response to the source.

to all of these sources are almost in phase at S, and as long as S is outside the advected region of heated fluid so that the positive spike in displacement in the heated fluid need not be considered. The localized pulse response at S has a vertical half-wavelength of $\pi U/N$, and varies in the horizontal as X^{-2}. Hence the distributed pulse response at S will be well approximated by (4) as long as

$$\Delta x \ll |X| = Ut, \quad (5a)$$

$$\Delta z \ll \pi U/N, \quad (5b)$$

i.e., if the Froude number $F = U/(N\Delta z) \gg \pi^{-1}$ and $t \gg \Delta x/U$.

These arguments can be substantiated by comparing them with LS's analytical calculations of the vertical displacement $\eta_{0dp}(t)$ ('dp' for distributed pulse) at S for a heating distribution $q_d(x, z) = q_0\{b/\pi(x^2 + b^2)\}\{H(d - |z|)/2d\}$. Here $H(\xi)$, the Heaviside step function, is 1 when $\xi \geq 0$ and 0 otherwise. This source is evenly distributed over a height range $-d < z < d$ and is at least half its peak strength over the range $-b < x < b$. Figure 2 shows $\eta_{0f}(t)$ as a function of time when $U/Nd = 1$. Taking $\Delta x = 2b$, one sees that for $t \geq 2\Delta x/U = 4b/U$, the response shows the t^{-1} decay predicted by (4) that is characteristic of the response to a localized pulse.

Now consider a distributed heat source $q_{dm}(x, z, t) = q_d(x, z)H(t)$ of finite size that is turned on at $t = 0$ and maintained. It may be regarded as a succession of very short heat pulses $q_d(x, z)\Delta\tau$ occurring at times 0, $\Delta\tau$, $2\Delta\tau$, ... The vertical displacement at S at time t due to the pulse that occurred at time τ is $\eta_{dpS}(T)$,

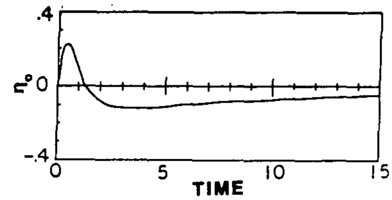

FIG. 2. The vertical displacement at S for SL's distributed heat source when $U/Nd = 1$, reproduced from their Fig. 3. Time is in units of b/U.

where $T = t - \tau$. The total vertical displacement $\eta_{dmS}(t)$ at S at time t due to the distributed maintained heat source is found by superposition:

$$\eta_{dmS}(t) = \int_0^t \eta_{dpS}(T)dT \approx \int_{\Delta x/U}^t \eta_{1pS}(T)dT$$
$$= -(q_0/2\pi NU^2)\log(Ut/\Delta x) \quad (t \gg \Delta x/U). \quad (6)$$

Here we have derived an approximation to the displacement at S by assuming that $\eta_{dmS}(T)$ is roughly equal to the localized pulse displacement $\eta_{1pS}(T)$ when $T > \Delta x/U$ and assuming that the contribution to the displacement from times T less than $\Delta x/U$, which is independent of t, is negligible by comparison. This will be true at large times t. The displacement grows large and negative at S. As pointed out in LS, the heating of air that is already increasingly warm is the source of energy for the growing response. If large negative displacements do not exist around S in the region of heating, then there will be no source of energy for large displacements elsewhere. Hence, in what follows we continue to concentrate on the displacement at the source position.

c. *Group velocity arguments and the response to a line buoyancy pulse*

So far we have largely recapitulated SL's line of reasoning showing that a maintained line buoyancy source leads to growing displacements. A crucial part of this analysis is that the displacement $\eta_{1pS}(t)$ at the position of a localized heating pulse only decays as fast as t^{-1} at large times, so that its integral diverges logarithmically. It is illuminating to use group velocity arguments (Lighthill, 1965) to explain this dependence. These arguments, which are the cornerstone of this paper, are very powerful, and they allow one to generalize SL's arguments by examining the pulse response in cases that the fluid is rotating or the heat source is three-dimensional, neither of which permits an analytic solution.

There are two linear hydrostatic gravity wave modes of a given wavenumber $\mathbf{k} = (k, m)$. They have frequencies $\omega \pm (\mathbf{k}) = Uk \pm Nk/m$ (Gill, 1982, p. 260) and group velocities

$$\mathbf{c}_g \pm (\mathbf{k}) = (c_{gx}^{\pm}, c_{gz}^{\pm}) = (U, 0) \pm (N/m, -Nk/m^2). \quad (7)$$

Let us call them the ω^+ and ω^- modes, respectively.

The Boussinesq hydrostatic wave energy density in physical space is (Gill, 1982, p. 140, with w neglected compared to u in the hydrostatic approximation):

$$E(\mathbf{x}, t) = \rho_0\{u^2/2 + B^2/2N^2\}. \quad (8)$$

The total wave energy is found by integrating E over physical space.

One can also define a spectral energy density in wavenumber space $E(\mathbf{k}, t)$. Let $f(x, z)$ be an arbitrary square-integrable function. Then f has a Fourier transform

$$f(k, m) = (2\pi)^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \exp(-ikx - imz) dx dz. \quad (9)$$

Parseval's equality (Sokolnikov and Redheffer, 1966, p. 72) states that

$$4\pi^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f|^2 dx dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(k, m)|^2 dk dm, \quad (10)$$

where Fourier transformed variables are italicized. Hence a reasonable definition of the spectral energy density $E(\mathbf{k}, t)$ is

$$E(\mathbf{k}, t) = \rho_0\{|u|^2 + |B|^2/N^2\}/8\pi^2. \quad (11)$$

The total wave energy at any time t can be found by integrating $E(\mathbf{k}, t)$ over all wavenumbers $\mathbf{k}$.

The energy spectrum (i.e., the dependence of $E(\mathbf{k}, t)$ on $\mathbf{k}$) due to a localized buoyancy pulse (1a) is particularly simple. Just after the heating at $t = 0^+$, the buoyancy has been increased impulsively at $x = z = 0$, but the fluid is still at rest: $B(\mathbf{x}, 0^+) = q_0\delta(x)\delta(z)$ and $u(\mathbf{x}, 0^+) = 0$. Hence (9) and (11) imply that $B(\mathbf{k}, 0^+) = q_0$ and $E(\mathbf{k}, 0^+) = \rho_0 q_0^2/8\pi^2N^2$ do not depend on $\mathbf{k}$ for this case. Since $u = 0$ initially, the spectral energy is evenly split between the $\omega^{\pm}$ modes with wavenumber $\mathbf{k}$:

$$E^{\pm}(\mathbf{k}) = E(\mathbf{k})/2 = \rho_0 q_0^2/16\pi^2N^2. \quad (12)$$

For linear waves in an unbounded homogeneous medium, spectral energy imparted to wavenumber $\mathbf{k}$ in each of these frequencies remains in that wavenumber and frequency. Thus the $E^{\pm}(\mathbf{k}, t)$ do not depend on time, and we will henceforth write the spectral energy as $E^{\pm}(\mathbf{k})$.

The key concept of dispersive wave theory is the wave packet, a region of wave disturbance which is confined in space within some characteristic distance ξ of a central position $\mathbf{x}_c(t)$ and is made up of waves whose wavenumbers do not differ from a carrier wavenumber $\mathbf{k}_c$ by more than a characteristic amount κ. In order that the wave packet have a well defined carrier wavenumber, ξ must be significantly larger than the carrier wavelength $\lambda_c = 2\pi/|\mathbf{k}_c|$. Wave packets are localized both in physical space and wavenumber (spectral) space. The distribution of spectral energy between wavenumbers in the wave packet is fixed, but the center of the wave packet moves at a speed $\mathbf{c}_g(\mathbf{k}_c)$ and the width of the wave packet increases due to dispersion of the waves in the packet as described below, so the spatial distribution of wave energy changes with time.

These concepts also apply to a localized pulse source of wave energy at a position S. The response to this source can be regarded as the superposition of a spectrum of wave packets of different carrier wavenumbers $\mathbf{k}_c$, each localized around S with a characteristic width equal to a few times $\lambda_c(\mathbf{k}_c)$. As time goes on, the wave packets separate, and one may regard the spectral energy in a wavenumber $\mathbf{k}_c$ as being spatially concentrated around $\mathbf{x}_c(t) = \mathbf{c}_g(\mathbf{k}_c)t$. This is nicely illustrated in Fig. 1. According to (5), the $\omega^{\pm}$ modes of wavenumber $\mathbf{k}_c = (k, m)$ should be observed at position $x_c = Ut \pm Nt/m$, $z_c = -\pm Nkt/m^2$. Consider the ω^- mode of wavenumber $\mathbf{k}_0 = (0, N/U)$, for instance. We find $x_c = z_c = 0$ because $\mathbf{c}_g(\mathbf{k}_0) = \mathbf{0}$. Indeed, near S the nodal lines of the displacement shown in Fig. 1 are approximately horizontal and separated by a vertical distance $\pi/m = \pi U/N$ as would be expected for a wave of wavenumber $\mathbf{k}_0$. The ω^+ mode of wavenumber $-\mathbf{k}_0$ also has group velocity zero and contributes to the displacement at S.

We can use the correspondence between wavenumber and position to calculate how the spectral energy in any fixed band of wavenumbers is spread over an increasing area of physical space by dispersion. Of particular interest to us is a wave packet with carrier wavenumber $\mathbf{k}_0$ with group velocity zero, since its energy will remain near S. Consider a wave packet made up of the ω^- modes in the upper stippled trapezoidal region in wavenumber space of Fig. 3a, centered around $\mathbf{k}_0$. If Δk and Δm are small, this region is almost rectangular and has an area $\Delta k \Delta m$. The above correspondence implies that the energy in these waves is spread over the region $R(t)$ in physical space shown in Fig. 3b and 3c at an earlier and a later time. The region $R(t)$ expands linearly with time in each direction due to the nonzero group velocity of waves on the borders of the trapezoid. If Δk and Δm are small, the area of R is $J^-\Delta k \Delta m$, where J^- is the Jacobian

$$J^-(\mathbf{k}_0, t) = |\partial(c_{gx}^- t, c_{gz}^- t)/\partial(k, m)|^{\mathbf{k}=\mathbf{k}_0}$$

$$= U^4t^2/N^2. \qquad (13)$$

Here R expands proportional to t in each direction (Fig. 3b, c), because each wave moves away from S at its group velocity.

To calculate the energy density in $R(t)$, one must also account for the ω^+ modes with wavenumbers near $-\mathbf{k}_0$ in the lower stippled trapezoid, which also have small group velocities and correspond to the same region $R(t)$ in physical space shown in Figs. 3b, c; an analogous calculation to (13) shows $J^-(-\mathbf{k}_0, t) = J^+(\mathbf{k}_0, t)$. The average energy density in $R(t)$ is the sum of contributions from the total energy in the two wave packets divided by the area of $R(t)$:

$$E_{1p}(\mathbf{0}, t) = E^+(\mathbf{k}_0)/J^+(\mathbf{k}_0, t) + E^-(-\mathbf{k}_0)/J^-(-\mathbf{k}_0, t)$$

$$= \rho_0 q_0^2/8\pi^2U^4t^2. \qquad (14)$$

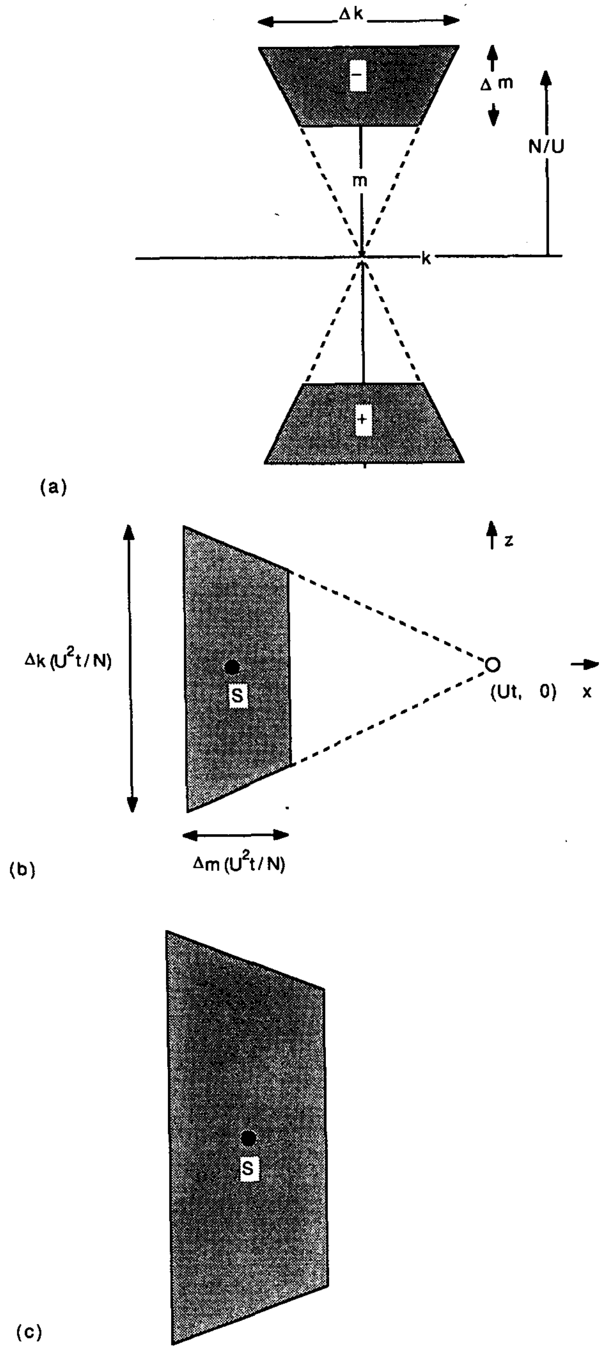

FIG. 3. The energy excited by a localized buoyancy pulse in the ω^- modes in the upper stippled trapezoid and in the ω^+ modes in the lower stippled trapezoid in wavenumber space in (a) is localized according to group velocity arguments in the stippled trapezoid in physical space around S in (b) at some time and in the stippled trapezoid in (c) at a later time.

By taking Δk and Δm to be small, $R(t)$ can be made to enclose an arbitrarily small area around S, so (14) will be taken to give the energy density at S.

It remains to translate this statement about the physical space energy density into a prediction of $\eta_{1pS}(t)$, the vertical displacement at S at a time t after a buoyancy pulse localized at S. By symmetry, the vertical velocity is an even function of z. Mass continuity then implies $u = 0$ along the x-axis and in particular at S. Thus, one can use (8) to recover the buoyancy at S from (14). The air which is at S at time t was at $x = -Ut$ at the time of the buoyancy pulse, so it was not heated at that time. For air which has at no time been heated, the buoyancy is due entirely to the vertical displacement, and $B(\mathbf{x}, t) = -N^2\eta(\mathbf{x}, t)$. Hence we obtain:

$$\eta_{1pS}(t) = -B_{1p}(\mathbf{0}, t)/N^2 = -\{2E_{1p}(\mathbf{0}, t)\}^{1/2}/N = -q_0/2\pi NU^2t. \quad (15)$$

It is important to recognize that while this result was for simplicity derived under the hydrostatic approximation, (15) will also hold for large times even if the hydrostatic approximation is not made because the response at S at large times relies only on the propagation characteristics of gravity waves with $k \ll m$, which are always hydrostatic. This can be verified by calculating the Jacobians (13) at $\mathbf{k} = \pm\mathbf{k}_0$ using the nonhydrostatic dispersion relation $\omega^{\pm}(\mathbf{k}) = Uk \pm Nk/(k^2 + m^2)^{1/2}$ (Gill, 1982, p. 260); they are identical to their hydrostatic counterparts.

The t^{-1} dependence of η is a geometrical effect, which relies only on the fact that there are wavenumbers with zero group velocity around which there is a finite rate of dispersion J. The small, but nonzero, group velocities of nearby wavenumbers spread their energy into a region in space that expands linearly with time in each direction, so that the energy density, a quadratic function of η, decreases as t^{-2}.

The logarithmically increasing response to a maintained buoyancy source does rely on one special feature of internal gravity waves. The buoyancy pulses that combine to make up the maintained source produce displacements that are all in phase at S, because the waves of zero group velocity also have zero frequency. In a system in which this is not true, such as barotropic Rossby waves in a uniform zonal current on a β-plane, an obstacle (or any other steady wave energy source) will produce a finite response; the pulse response (corresponding to the obstacle being present only for a short interval that starts at a time t before the response is observed) still produces displacements of order t^{-1}, but they do not add in phase to produce a divergent response to a steady obstacle.

3. Distributed heat sources, boundaries and vertical inhomogeneities

In this section we will characterize the heat source distributions which, when steadily maintained, lead to a steady state response. Smith and Lin pointed out that a steady response will occur if the horizontally integrated heating is zero. In the appendix, we show that this is a special case of a more general criterion: If a buoyancy source $q_m(x, z)H(t)$ is turned on at $t = 0$, then a finite, steady displacement field $\eta(x, z)$ will set up only when

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdzq_m(x, z)\exp(\pm iNz/U) = 0; \quad (16)$$

i.e., when there is no projection of the heat source on the wavenumbers $\pm\mathbf{k}_0 = (0, \pm N/U)$ corresponding to the ω^+ and ω^- modes which have zero group velocity.

This is physically easy to understand. After a pulse of heating, the displacement at any fixed position is ultimately dominated by the ω^+ modes near $\exp(-i\mathbf{k}_0\cdot\mathbf{x}) = \exp(-iNz/U)$, and the ω^- modes near $\exp(i\mathbf{k}_0\cdot\mathbf{x}) = \exp(iNz/U)$, which have very small group velocities. A maintained source, or train of pulses, is continuously pumping energy into these modes unless (16) holds. In two dimensions, dispersion cannot spread this energy fast enough to prevent the logarithmic buildup of displacement with time.

The effect of horizontal boundaries on the response when U and N are uniform can be found by using the method of images. A single boundary, at $z = 0$ for instance, below a heat source $q_m(x, z)$, $z > 0$, can be replaced by a source $q_m(x, -z) = -q_m(x, z)$. The steadiness of the response can then be diagnosed from (16). In general, the addition of the image source does not make the projection zero unless it would have been zero without the lower boundary, but some pathological cases can be found. For example, a source $q_m(x, z) = g(\mathbf{x})\delta(z - \pi U/2N)$, where $g(x)$ is any positive integrable function of x, will produce a steady response when a boundary is placed at $z = 0$, but in an unbounded fluid the response will grow with time.

Two boundaries a distance H apart have a much more striking effect on the response. They can be replaced by an image source distribution in which q_m is extended from $0 < z < H$ to all space by the relations $q_m(x, z + 2jH) = q_m(x, z)$ and $q_m(x, 2jH - z) = -q_m(x, z)$, $j = 0, \pm1, \pm2, \cdots$. The vertical wavenumbers $m_n = n\pi/H$ in this extended heat source are quantized, so the steady state criterion (16) will be satisfied unless $N/U = n\pi/H$ for some integer n, when there is a normal mode of zero frequency that can be forced to resonate by the steady heating.

We show in the appendix that a similar criterion to (16) holds even when N and U vary with z, as long as there is a continuous spectrum of normal modes of zero frequency [such as $\exp(imz)$, where m is arbitrary and can be continuously varied, in the case of constant N and U]. This will be the case if the Scorer parameter $S = N^2/U^2 - U_{zz}/U$ becomes positive and uniform as $z \to \pm\infty$. The criterion holds even if there are critical levels $U = 0$ for stationary waves.

If $S < 0$ as $z \to \pm\infty$, then stationary waves are trapped, and there is a discrete spectrum of steady modes. The response will be steady unless there happens to be a mode of zero horizontal group velocity, in which case a resonance occurs that is similar to that which can be set up by two boundaries.

4. Mass sources

Raymond (1983) pointed out that in the hydrostatic approximation, heat sources and mass sources have equivalent effects on the fluid outside the source region. The linear response to a buoyancy source $q(x, z, t)$ in a stratified fluid moving with an ambient horizontal speed U is governed by (1a–d). If w is partitioned as $w = w_a + q/N^2$, then the buoyancy and continuity equations can be written in terms of the 'adiabatic' vertical velocity w_a:

$$B_t + UB_x + N^2 w_a = 0, \tag{17a}$$

$$u_x + w_{az} = M = -q_z/N^2. \tag{17b}$$

In the hydrostatic approximation, w does not appear in the vertical momentum equation. If w_a is identified as the vertical velocity, then (1a, b) and (17a, b) are the linear hydrostatic equations of motion given an equivalent mass source $M = -q_z/N^2$.

Raymond (1983) showed that the response to a localized pulse mass source $M(x, z, t) = M_0\delta(x)\delta(z)\delta(t)$ is

$$\eta_{1pM}(x, z, t) = (M_0/2\pi X)\sin(Nzt/X), \tag{18a}$$

$$u_{1pM}(x, z, t) = (M_0 N/2\pi X)\cos(Nzt/X). \tag{18b}$$

At the original source position $x = 0$ ($X = -Ut$), $u_{1pMS} = -M_0N/2\pi Ut$. If the source is of finite width, then the deceleration will be reduced at times less than or comparable to $\Delta x/U$, but will be unchanged at large times. This mass source is equivalent to a buoyancy source $q_M(x, z, t) = -N^2M_0\delta(x)H(z)\delta(t)$. Except in the heated fluid ($X = 0, z > 0$), $w_a = w$, so the displacement due to the heat source q_M should be the same as the displacement (18a) due to the equivalent mass source. Since $\partial q_M/\partial z = q_{1p}(x, z, t)$, we should therefore recover the displacement due to a point buoyancy impulse for nonzero X by differentiating η_M with respect to z. Comparison with (3) shows that this is indeed true. We can therefore anticipate that our group velocity arguments about heat sources should be generalizable to appropriate mass sources.

A mass source turned on at time zero and maintained can be regarded as a sequence of mass pulses injected into the flow at successive times $\tau > 0$. By superposition of the $u_{1pMS}(t - \tau)$ from $0 < \tau < t$, it is clear that at the source position there is an increasing horizontal velocity deficit proportional to log(t). Physically, the energy source for these perturbations is the relatively high pressure at the mass source, which makes the injected mass do work on the fluid.

A sufficient condition that a distributed mass source $M_{md}(x, z)$ can produce a steady response can be found by finding an equivalent heat source such that $M_{md} = -q_{mdz}/N^2$ and then applying the criterion (16). After one integration of (16) by parts in z, we obtain two terms, a boundary term which is the x integral of $q_{md} \times \exp(\pm iNz/U)$ evaluated between the limits $\pm\infty$ in z, and an area integral. The boundary term is indeterminate unless the x integral of $q_{md} \to 0$ as $z \to \pm\infty$, in which case it vanishes. This condition translates to the condition that there be no net mass source integrated over all space:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdz M_{md}(x, z) = 0; \tag{19a}$$

The remaining integral from the integral by parts must also vanish to satisfy (16), so we also require

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdz M_{md}(x, z)\exp(\pm iNz/U) = 0. \tag{19b}$$

If both (19a) and (19b) are satisfied, the response due to the maintained mass source M_{md} will approach a steady state (although these conditions may not be necessary).

Two physical problems which can be modeled by mass sources in moving stratified flows are the flow up over a ramp or plateau and the flow due to a density current. Suppose that a moving stratified fluid with a horizontal velocity far upstream equal to U flows up over a bottom surface $z = h(x)$ such that $h \to 0$ as $x \to -\infty$ and $h \to h_0$ as $x \to \infty$. If h_0 is small, then the boundary condition can be linearized to the condition that $w = Uh_x$ at $z = 0$. A steady mass source $Uh_x(x)\delta(z)$ distributed along a flat bottom ($z = 0$) produces an identical response.

The response given by (18a, b) applies to a heat source in a fluid unbounded in all directions. Consider a mass source $2Uh_x(x)\delta(z)$ (of integrated strength $M_0 = 2Uh_0$) in an unbounded fluid. By symmetry, it causes no vertical displacements on the line $z = 0$. Thus it automatically satisfies the boundary condition $w = 0$ on $z = 0$. Furthermore, half of the injected mass ends up above $z = 0$, reproducing the "plateau" mass source considered in the previous paragraph.

Equation (19a) suggests that the flow will not be steady. Indeed, if the plateau suddenly grew up at time zero out of the surface $z = 0$, then the flow would increasingly decelerate as it approached the plateau to a minimum speed $u_{\min}(t)$ in the middle of the windward slope of the plateau which is found by using (18b), corrected for the finite source width Δx:

$$u_{\min}(t) \approx \int_{\Delta x/U}^{t} u_{1pMS}(\tau)d\tau$$

$$= -(Nh_0/\pi)\log(Ut/\Delta x). \tag{20}$$

When $u_{\min}(t)$ becomes comparable to $-U$, nonlinear effects will become important, possibly leading to the formation of a stagnant region upstream of the stationary mass source.

Benjamin (1968) showed that a steady inviscid density current due to the flow of a shallow homogeneous layer of fluid of density $\rho + \Delta\rho$ into a homogeneous fluid that has a lesser density ρ is impossible, due to the impossibility of simultaneously conserving energy and momentum. One might wonder whether this conclusion was still correct if the less dense fluid were stratified and gravity waves produced by the density current could correct the momentum imbalance. Viewed in a frame moving with the density current, the denser fluid is just a (deformable) plateau-shaped obstacle to the oncoming flow, so the argument above shows that a linear steady state will not be achieved in response to a shallow density current even if the less dense fluid is stratified.

It is interesting to estimate how fast the displacements and decelerations should grow when a density current with a quasisteady nose profile moves into a stratified atmosphere, assuming that an insignificant fraction of the ambient air is entrained into the density current. Benjamin's (1968) formula for the speed of a density current moving into stagnant air is $U = (2gh_0\Delta\rho/\rho)^{1/2}$. Taking $\Delta\rho/\rho = 0.05$ and assuming that the density current has a depth $h_0 = 1000$ m behind its nose, we obtain $U = 10$ m s^{-1}. The density current is equivalent to a mass source $M_0 = 2Uh_0 = 20\,000$ m^2 s^{-1} in an unbounded fluid. If $N = 10^{-2}$ s^{-1} and surface air rises to the height h_0 in a horizontal distance $\Delta x = 1000$ m, the horizontal velocity at the nose is decelerated to 10 m s^{-1} $-$ 3.2 log$\{t/(100\text{ s})\}$ according to (20). This formula predicts stagnation of surface air ahead of the nose roughly twenty minutes after the formation of the density current; the nonlinear development of gravity waves in the stable layer and/or entrainment into the head of the density current must be crucial to the further development of the flow and the force balance and shape of the nose of the density current.

5. The effects of the third dimension and of rotation

The infinite response to a localized maintained heat source is specific to two-dimensional flow in a nonrotating atmosphere. In this section, we use group velocity arguments to estimate the displacement field in response to a buoyancy source that is localized in both horizontal directions and a line buoyancy sources in a rotating fluid. These estimates show that in both of these cases, a maintained buoyancy source produces a displacement field which tends toward a steady state.

a. Three-dimensional heat sources

Let $\eta_{dm3}(x, y, z, t)$ be the displacement field at time t caused by a maintained distributed buoyancy source $q_{3D}(x, y, z)H(t)$ centered at the origin S with characteristic length Δx, width Δy and height Δz. As in section 2, this can be regarded as the superposition of the displacements $\eta_{dp3}(x, y, z, T)$, where $T = t - \tau$, due to a succession of distributed buoyancy pulses $q_{3D}(x, y, z) \times \delta(t - \tau)$ at times $0 < \tau > t$. In particular, the displacement at S due to the maintained source is

$$\eta_{dm3S}(t) = \int_0^t \eta_{dp3S}(T)dT. \qquad (21)$$

If the pulse displacement $\eta_{dp3S}(T)$ at S decays faster than T^{-1} at large times, the integral has a finite limit at large times and a steady state is approached. Suppose that $T \gg \Delta x/U$, $\Delta y/U$ and the source Froude number is large. As shown in section 2, the hydrostatic approximation can be introduced without causing significant errors in $\eta_{dp3S}(T)$; this is desirable since it simplifies the calculations involving the dispersion relation. Furthermore, the arguments of section 2 show that $\eta_{dp3S}(T)$ is approximately equal to the displacement $\eta_{1p3S}(T)$ in the hydrostatic approximation due to a localized buoyancy pulse $q_0\delta(x)\delta(y)\delta(z)\delta(t)$ whose integrated buoyancy input is the same as that of $q_{3D}(x, y, z)\delta(t)$. Hence, it suffices to estimate $\eta_{1p3S}(T)$ at large times T.

Group velocity arguments similar to those of section 2 show that $\eta_{1p3S}(t)$ decays faster than $t^{-3/2}$ for large times. The two hydrostatic gravity wave modes have dispersion relations $\omega^{\pm}(\mathbf{k}) = Uk \pm N\kappa/m$ and group velocities

$$\mathbf{c}_g^{\pm}(\mathbf{k}) = (U, 0, 0) \pm (Nk/\kappa m, Nl/\kappa m, -N\kappa/m^2), \qquad (21)$$

where $\mathbf{k} = (k, l, m)$ and $\kappa = (k^2 + l^2)^{1/2}$. The ω^- modes with wavenumbers close to $\mathbf{k}_0 = (0, 0, N/U)$ and ω^+ modes with wavenumbers close to $-\mathbf{k}_0$ have $\mathbf{c}_g \approx \mathbf{0}$, if k/m is positive and l/k is small.

The energy density $E_S(t)$ at S can be calculated just as in section 2. It is due to the modes with very small group velocities. The ω^- modes with wavenumbers close to $\mathbf{k}_0 = (0, 0, N/U)$ and ω^+ modes with wavenumbers close to $-\mathbf{k}_0$ have $\mathbf{c}_g \approx \mathbf{0}$, if k/m is positive and l/k is small.

The spectral energy put into each mode by the buoyancy pulse at $t = 0$ can be found as in the two dimensional case. At $t = 0^+$, $u(\mathbf{k}) = v, (\mathbf{k}) = 0$ and $b(\mathbf{k}) = q_0$. The left side of Parseval's equality (7b) and hence the denominator of the definition (8) of the spectral energy density have additional factors of 2π due to the additional dimension, so that $E(\mathbf{k}) = \{|u|^2 + |v|^2 + (|b|^2/N^2)\}/16\pi^3$. Evaluating this expression at $t = 0^+$ and splitting the spectral energy symmetrically between the two modes to satisfy the condition of no velocity at $t = 0^+$, we obtain $E^-(\mathbf{k}) = E^+(-\mathbf{k}) = q_0^2/32\pi^3N^2$.

The spectral energy in the ω^- modes whose wavenumbers lie in some volume dV centered on $\mathbf{k}_0 = (0, 0, N/U)$ is $E^-(\mathbf{k})dV$. According to group velocity arguments, these wavenumbers will mainly be localized to a region in physical space whose volume is $dV = J_{3D}^-(\mathbf{k}_0, t)dV$, where

$$J_{3D}^-(\mathbf{k}_0, t) = \partial(c_{gx}^-t, c_{gy}^-t, c_{gz}^-t)/\partial(k, l, m)|^{\mathbf{k}=\mathbf{k}_0}$$
$$= N^3t^3/\kappa m^5|^{\mathbf{k}=\mathbf{k}_0} = \infty, \qquad (23)$$

so that the energy density in physical space E_S^- at S due to the ω^- modes is $E^-(\mathbf{k}_0)/J_{3D}^-(\mathbf{k}_0, t) = 0$. A similar argument holds for the ω^+ modes. This should be interpreted as follows: The dispersion of modes whose wavenumbers are near $\pm\mathbf{k}_0$ is very large; depending on the ratio of the small quantities k and l, $\mathbf{c}_g$ can lie anywhere on a horizontal ring of radius $N/m = U$ centered at $(U, 0, 0)$. Only the small fraction of these modes with $\mathbf{c}_g$ close to zero contribute to the displacement at S. Even if $J_{3D}^{\pm}$ were finite, E_S would decay like t^{-3}. The extreme dispersion implies E_S decays faster than t^{-3}. By symmetry, there are no horizontal velocities on the plane $z = 0$; therefore the three-dimensional analogue of (8) implies $\eta_{1p3S}(t) = \{2E_S(t)/N^2\}^{1/2}$, which decays faster than $t^{-3/2}$ as claimed. A rather difficult asymptotic analysis, available from the author as a technical report, shows that in fact $\eta_{1p3S}(t) = -q_0/(4\pi NU^3t^2)$ for large time.

Since $\eta_{1p3S}(t)$ decreases faster than t^{-1} at large time, a maintained three-dimensional heat source does ultimately force a steady response. The fundamental reason for the rapid decrease in the displacement after a three dimensional buoyancy pulse is the dispersion of energy in the y (transverse) direction, which reduces the energy density at S.

An elongated ($\Delta y \gg \Delta x$) maintained buoyancy source $q_{3D}(x, y, z)H(t)$ can induce quite large vertical displacements at the center S of the source. Buoyancy pulses from early times τ such that $T = t - \tau \gg \Delta y/U$ produce very small displacements because of dispersion in the transverse direction, as argued above. However, those pulses with $\Delta x/U < T < \Delta y/U$ can be expected to produce displacements at S similar to the displacement $\eta_{m1S}(T)$ appropriate to a localized maintained line source, if q_0 is replaced in Eq. (12) by a cross sectional x–z integral of q_{3D} at $y = 0$. This can be understood by dividing up the elongated source into a number of sources, each of which has a breadth Δx. Let R be one of these sources from one end of the original elongated source. A buoyancy pulse at R at a time τ causes a significant response at S at those times $\tau + T$ when a significant fraction of the waves with $c_{gx} = c_{gz} = 0$ ($k, l \ll m = \pm N\kappa/Uk$) generated by R have reached S. These waves have $c_{gy} = Nl/\kappa m = U(l/k)$. Since wave energy is equally distributed among the small wavelengths, l/k is O(1) and c_{gy} is O(U) for most of the wave energy reaching S. Thus it takes a time $T = O(\Delta y/2U)$ for the wave energy from a pulse at R to significantly affect the displacement at S. At times $T \ll \Delta y/2U$, the response at S would only be weakly affected if additional sources with $y > \Delta y/2$ were added onto the end of the line; the response at S to pulses of buoyancy from the elongated source at such times T is nearly the same as if the source were infinitely long. The lower cutoff on T is caused by the finite source width Δx as discussed in the two-dimensional case. The steady state displacement due to a maintained elongated source will be dominated by the logarithmically large contribution from the times when the response is well approximated by the displacement due to a localized line buoyancy pulse:

$$\eta_{3D}(t \to \infty) \approx \int_{\Delta x/U}^{\Delta y/U} \eta_{m1S}(T)dT = q_0/2\pi NU^2 \log(\Delta y/\Delta x) \quad (\Delta y \gg \Delta x). \quad (26)$$

As the source becomes more line-like, the downward steady state displacements near the source become increasingly large.

b. A line heat source in a rotating environment

Consider a localized line buoyancy pulse at the origin S in a fluid moving with speed U relative to a reference frame rotating about the vertical with Coriolis parameter f. Due to the rotation, there are no inertia-gravity waves in the fluid whose group velocities are zero. Some waves have group velocities which are arbitrarily close to zero, so that their effects are felt at S for some time, but rotation causes energy in all waves with $c_{gz} = 0$ to be advected downstream with the fluid. Hence, the displacement $\eta_{1prS}(T)$ at S due to a localized buoyancy pulse a time T before the observation time decays more rapidly than T^{-1}, and the response to a maintained source is finite.

In the hydrostatic approximation, the dispersion relations and group velocities of the two gravity wave modes of wavenumber $\mathbf{k}$ are

$$\omega^{\pm}(\mathbf{k}) = Uk \pm Nk/\alpha m, \quad (27)$$

$$\mathbf{c}_g^{\pm}(\mathbf{k}) = \mathbf{U} \pm (N\alpha/m, -N\alpha k/m^2), \quad (28)$$

$$\alpha = \{1 + (fm/Nk)^2\}^{-1/2}. \quad (29)$$

The group velocities of the waves relative to the mean flow $\mathbf{U} = (U, 0)$ are smaller than in the nonrotating case by a factor α, which approaches zero for waves with $k \ll m$ that have almost horizontal phase surfaces. There are no waves with $\mathbf{c}_g = \mathbf{0}$, although there are waves with $k \ll m$ for which $\mathbf{c}_g$ is arbitrarily close to zero. At sufficiently large times, group velocity arguments suggest that the $\omega^{\pm}$ modes with wavenumber $\mathbf{k}$ will be observed at $\mathbf{x} = \mathbf{c}_g^{\pm}(\mathbf{k})t$, or from (28),

$$x = Ut \pm N\alpha t/m, \quad z = \pm(-N\alpha kt/m^2). \quad (30a)$$

Equation (30a) can be inverted to determine the wavenumbers of the ω^- and the ω^+ modes which will be observed at some fixed position $\mathbf{x}$ at time t. Let $X = x - Ut$. We first note that $k/m = -z/X$, so that $\alpha = \{1 + (fX/Nz)^2\}^{-1/2}$; we can then find m and lastly find k:

$$\mathbf{k}^{\pm}(\mathbf{x}, t) = \pm(-N\alpha zt/X^2, N\alpha t/X). \quad (30b)$$

Figure 4 illustrates the wave pattern produced by a localized buoyancy pulse $q_0\delta(x)\delta(y)\delta(z)$ in a rotating fluid at a time comparable to f^{-1} as deduced from (30b). The dashed nodal lines of zero vertical displacement correspond to places where the phases $\phi^{\pm} = k^{\pm}x$

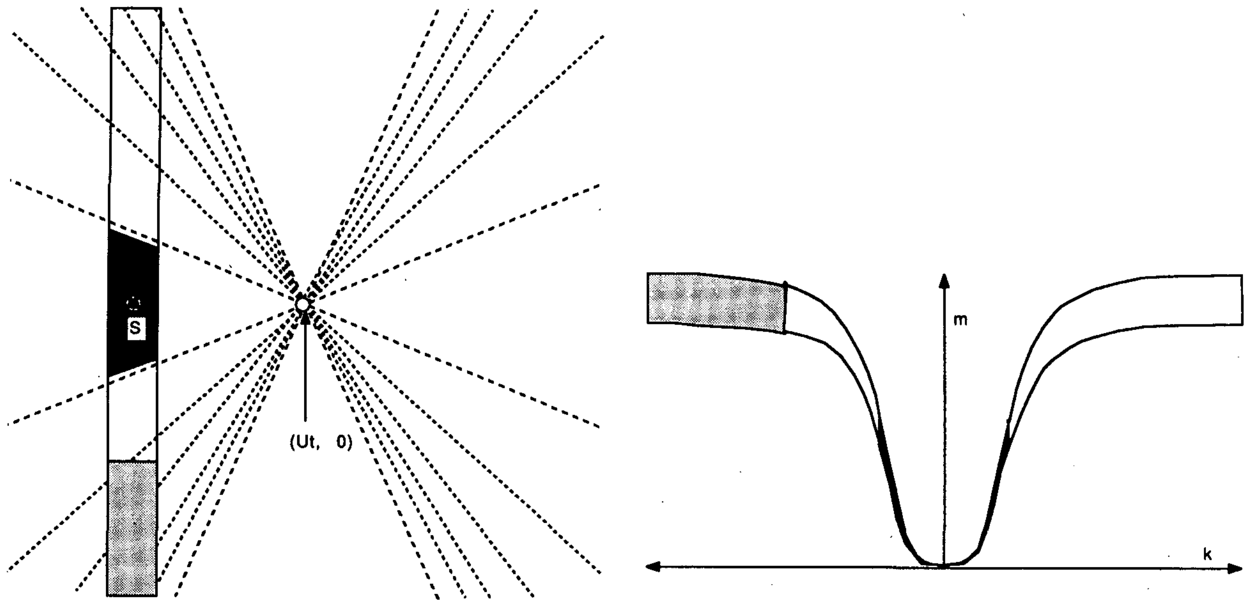

FIG. 4. (a) Lines of zero vertical displacement for a point impulsive heat source in a moving, rotating, stratified fluid (dashed). The lightly and heavily stippled boxes in physical space correspond according to group velocity arguments with the similarly demarcated regions in wavenumber space in Fig. 4(b).

$+ m^{\pm}z - \omega^{\pm}t$ of the displacements produced at position (x, z) by the ω^+ and the ω^- modes differ by an odd multiple of π. Using (27) and (30b), one finds the phase difference $\phi^+ - \phi^-$ to be $2Nzt/\alpha X$; nodal lines occur where $Nzt/\alpha X = (n + 1/2)\pi$ for some integer n. For $Nz/fX \gg 1$, the vertical spacing between nodal lines is $\pi X/Nt$ as in the nonrotating case; for $Nz/fX \ll 1$, nodal lines become more widely spaced. Physically, the horizontal group velocity of low-frequency waves with a given vertical wavenumber relative to the mean flow is retarded compared to the nonrotating case, so the vertical wavenumber of the waves observed at any fixed X at some time must be smaller to compensate, leading to an increased spacing between nodal lines as z becomes small.

As z approaches 0, the dominant $\omega^{\pm}$ modes become purely inertial oscillations with frequencies $\pm f\,\mathrm{sgn}(z/X)$ and phase difference $2ft\,\mathrm{sgn}(z/X)$.

The energy density at S can be again predicted by group velocity arguments as in section 2. The spectral energy density in each mode produced by a localized buoyancy pulse is $E^{\pm}(\mathbf{k}) = q_0^2/8\pi^2N^2$ as in the nonrotating case, because the presence of the Coriolis force does not change the expression (6) for the energy density (Gill, 1982, p. 266) and hence also does not change the expression (8) for the spectral energy density. The spatial energy density $E(\mathbf{x}, t) = \{E^+(\mathbf{k}^+)/|J_r^+(\mathbf{k}^+)| + E^-(\mathbf{k}^-)/|J_r^-(\mathbf{k}^-)|\}$. The Jacobians $J_r^{\pm}(\mathbf{k}) = \partial(c_{gx}^{\pm}t, c_{gz}^{\pm}t)/\partial(k, m)|_{\mathbf{k}(x,t)} = -N^2\alpha^2/(m^{\pm})^4$. When the Jacobians are evaluated for the wavenumbers $\mathbf{k}^{\pm}(0, z, t)$ which are found at $x = 0$ and some height z, we find $J_r^{\pm}(\mathbf{k}^{\pm}) = U^4t^2/N^2\alpha^2$ and $E(0, z, t) = q_0^2\alpha^2/8\pi^2U^4t^2$. At S, where $\alpha = z = 0$, these arguments predict that the energy density $E_S(t)$ (and thus the displacement) is zero.

Figure 4b illustrates geometrically why the energy density is small near S. According to (30b), the lightly stippled rectangle of Fig. (4a) will contain waves whose wavenumbers lie in the corresponding region in Fig. (4b). Since $fX/Nz > 1$ in this rectangle, its corresponding area in wavenumber space is nearly the same as if there were no rotation. The waves which fill the heavily stippled region of Fig. 4a surrounding S occupy a much smaller region in wavenumber space (heavily stippled in Fig. 4b) and hence produce a smaller energy density in physical space.

Group velocity arguments correctly predict the local wavelength and energy density only where these quantities or varying slowly compared to a local wavelength. As $z \to 0$, (30b) implies $\lambda(\mathbf{x}, t) = 2\pi/|\mathbf{k}(\mathbf{x}, t)| \to \infty$ since $\alpha \to 0$, so group velocity arguments will locally break down. We interpret the prediction that $E_S = 0$ to actually mean that $E_S(t)$ decays faster than t^{-2} for large times. The displacement $\eta_{1prS}(t)$ is no larger in magnitude than $(2\langle E_S(t)\rangle)^{1/2}/N$; thus $\eta_{1prS}(t)$ decays faster than t^{-1} as claimed. In fact, direct evaluation of the Fourier transform integrals shows that when $ft \gg 1$, $\eta_{1prS}(t) = -q_0 \cos(ft - \pi/4)/(2^{1/2}\pi^{3/2}NU^2f^{1/2}t^{3/2})$, an inertial oscillation decaying like $t^{-3/2}$. This calculation is also available from the author as part of a technical report.

For times $ft \ll 1$, the region $|z| < fUt/N$ of diminished group velocities cover such a small fraction of a vertical wavelength N/U that the effect of rotation on the buoyancy pulse response is negligible.

Consider a maintained buoyancy source (i.e., a series of buoyancy pulses) in a rotating fluid. The rapid decay and the oscillatory form of the displacement at S due to buoyancy pulses from times $T = O(f^{-1})$ prior to the observation time imply that an approximately steady displacement at S is reached within a time $O(f^{-1})$.

The smaller the characteristic width Δx of the source, the more the displacement at S can build up due to the responses to buoyancy pulses from times $\Delta x/U < T < f^{-1}$ prior to the observation time when the pulse response is not reduced by the finite source size or by rotation. A crude estimate of the steady state displacement at S can be found by integrating the two-dimensional localized pulse response from (15) over these times:

$$\eta_{1prS}(t \to \infty) \approx \int_{\Delta x/U}^{1/f} \eta_{1pS}(T)dT = (q_0/2\pi NU^2)\log(U/f\Delta x). \quad (34)$$

To summarize, the response to both rotating and three-dimensional heat pulses can be clearly understood using group velocity arguments. In contrast to the two-dimensional nonrotating case the displacement at the original source position S diminishes faster than t^{-1} at large times in both cases, permitting a steady displacement field to form in response to a maintained heat source. For a three-dimensional source, the reduced response is due to strong dispersion of wave energy in the direction transverse to the flow. For a rotating source, it is due to the reduced upstream propagation of the energy relative to the mean flow in the low-frequency waves that dominate the response at S.

The importance of rotation and three-dimensionality can be estimated for some physically interesting cases. Smith and Lin (1982) investigated the effect of the latent heating produced by orographic rain. In LS, an example was given with $U = 10$ m s^{-1}, $N = 10^{-2}$ s^{-1}, and a rainfall rate of $r = 2.5$ mm h^{-1} over a width $\Delta x = 40$ km on the windward side of the mountain, which produces buoyancy at a rate $q_0 = (g\rho_w L/\rho_0 C_p T_0)r\Delta x = 2300$ m^3 s^{-3}, where L is the latent heat of vaporization of water vapor, ρ_w/ρ_0 is the ratio of the density of liquid water to a reference air density, and the remaining symbols were defined in section 2. Without rotation or three-dimensionality, the downward displacement $\eta_{dmS}(t)$ at S due to the heating alone (which must be superposed on the displacements forced by the orography) can be estimated from (6). This implies $\eta_{dmS}(t) \approx -366 \log\{t/(4000 \text{ s})\}m$, in good qualitative agreement with SL's numerical results at $t = 13\,000$ s. As pointed out by SL, the heating induced subsidence may significantly reduce the rainfall rate (and hence the heating) at later times. According to (26), the three dimensionality of the source suppresses the growth of the displacement after a time $t = \Delta y/U$. If we consider a mountain range of breadth $\Delta y = 300$ km, the displacement should build up until 30 000 s, resulting in a steady downward displacement on the order of 750 m. According to (34), rotation limits the downward displacement after a time $t = f^{-1} = 10\,000$ s to about 350 m.

6. Conclusions

A maintained line buoyancy source in a steadily moving stratified fluid creates a logarithmically growing response after it is turned on. The response can be understood by thinking of the source as a train of buoyancy pulses. A pulse from a time T before the observation time t causes a downward displacement proportional to T^{-1} at the original heating position; these displacements add to create the growing response.

The main thrust of the present paper has been to use group velocity arguments to understand the pulse response for the two-dimensional line sources and then to generalize these arguments to construct the response to two-dimensional line buoyancy pulses in a rotating fluid, three-dimensional pulses, and line mass pulses. Superposition then allows us to determine whether a maintained heat or mass source ultimately produces a steady state response in these cases. The response to a mass pulse again decays as T^{-1}, so steady linear flow over a maintained effective line mass source such as a ramp-shaped mountain or a steady inviscid shallow density current cannot occur in a uniformly stratified fluid. However, rotation or three-dimensionality causes the displacement at the original position of a buoyancy pulse to decay faster than T^{-1} and allows a steady state to set up in response to a maintained source. The amplitude of the steady state displacements at the position of a maintained source are $O(\log(U/f\Delta x))$ for a line source of width Δx in a rotating frame and $O(\log(\Delta y/\Delta x))$ for an elongated three-dimensional source of width Δx and breadth Δy.

The displacement field near the original position of a line buoyancy or mass pulse at large time T is dominated by wavenumbers $\mathbf{k} = (0, \pm N/U)$ of very small group velocity. The dispersion of these wavenumbers causes the energy density at S to decrease as T^{-2} in two dimensions. This is reflected in a T^{-1} decrease of the displacement η. Since the frequency of these modes is close to zero, the displacements due to pulses at different times are in phase, forcing the displacement to build up. Waves from a three-dimensional pulse disperse in the transverse direction also, so the displacement decreases faster than T^{-1}. In a rotating frame there are no gravity waves of precisely zero group velocity relative to the fixed frame, so the displacement again decreases faster than T^{-1} at large times.

A steady state response to a maintained distributed buoyancy source is possible if the buoyancy source has no projection on the modes of zero group velocity, so that very little energy will be pumped into the nearby modes of small group velocity, preventing the buildup of the displacement. This generalizes SL's criterion for a steady state that the horizontally averaged heating at any level should be zero.

The potential nonlinear effects due to the growing displacement field would be a quite intriguing subject for further analytical and numerical investigation. Two nonlinear processes that could stop the growth of the displacement field are wave reflection due to modification of the stability and velocity fields by wave steepening and breaking, and downward displacements at the heat source S on the order of a quarter wavelength $\pi U/2N$ of the standing wave of zero group velocity, which would cause the fluid moving into the heated region to not necessarily have experienced the full effect of the downward displacements at the level of the buoyancy source, because this fluid started too far above this level.

In the orographic rain example of section 5, rotation limits the downward displacement at the position of the heat source well before it reaches the quarter-wavelength threshold $\pi U/2N \approx 1500$ m at which nonlinear effects must become important. But in the density current example of section 4, neither rotation or three-dimensionality can prevent the rapid buildup of decelerations and vertical displacements ahead of the nose and nonlinear effects must become important. Suitably interpreted, group velocity arguments should prove their utility as a tool for understanding even the nonlinear response of the atmosphere to a localized internal forcing.

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APPENDIX

Criteria for a Steady Response to a Distributed Maintained Heat Source

We look for steady solutions of the Taylor-Goldstein equation including heating $q(x, z)$ when $U(z)$ and $N^2(z)$ may vary with height. Let $S(z) = N^2/U^2 - U_{zz}/U$ be the Scorer parameter, and let $w(k, z)$ be the Fourier transform of $w(x, z)$ in the x direction. Then

$$U^2\{w_{zz} + S(z)w\} = q(x, z). \qquad \text{(A1)}$$

Here, $w(k, z)$ obeys the radiation conditions $w \propto \exp\{i\mu \pm |z| \operatorname{sgn}(k)\}$ as $z \to \pm\infty$. We are assuming that as $z \to \pm\infty$, N and U tend to constant asymptotic values, so that $\mu_\pm = \lim_{z\to\pm\infty} S^{1/2}(z)$ are well-defined and positive. The displacement $\eta(x, z)$ from the upstream conditions is required to remain finite at all heights, i.e., $\eta(x, z)$ is bounded as $x \to \infty$. Equivalently,

$$Uw(0, z) = \int_{-\infty}^{\infty} Uw(x, z)dx = \eta(\infty, z) \quad \text{is finite.}$$

The Green's function for (A1) given these boundary conditions is

$$G(z, z'; k) = \begin{cases} G_+(z, z') = c_+\phi_{1+}(z_>)\phi_{2+}(z_<), & k > 0 \\ G_-(z, z') = c_-\phi_{1-}(z_>)\phi_{2-}(z_<), & k < 0. \end{cases} \qquad \text{(A2)}$$

$z_>$ and $z_<$ are the larger and smaller of z and z', respectively. $\phi_{1\pm}$ and $\phi_{2\pm}$ are homogeneous solutions of (A1) with $\phi_{1\pm}(z) \sim \exp\{\pm i\mu_+ z\}$ as $z \to \infty$ and $\phi_{2\pm}(z) \sim \exp\{\pm i\mu_- z\}$ as $z \to -\infty$. The $c_\pm$ are normalization constants. $w(k, z)$ can be written in terms of the Green's function:

$$w(k, z) = \int_{-\infty}^{\infty} dz' G_\pm(z, z')q(k, z') \begin{pmatrix} +: k > 0 \\ -: k < 0 \end{pmatrix}. \qquad \text{(A3)}$$

If $w(0, z)$ is to be finite, then

$$\lim_{k\to0^+} w(k, z) = \lim_{k\to0^-} w(k, z) = w(0, z) < \infty.$$

It is useful to examine this condition in the region above the heating at which N and U have reached their asymptotic values for large z. Then, since $z > z'$ in (A3),

$$\lim_{k\to0^+} w(k, z) = c_+\phi_{1+}(z) \int_{-\infty}^{\infty} dz'\phi_{2+}(z')q(0, z'),$$

$$\lim_{k\to0^-} w(k, z) = c_-\phi_{1-}(z) \int_{-\infty}^{\infty} dz'\phi_{2-}(z')q(0, z'). \qquad \text{(A4)}$$

At large z, $\phi_{1+}(z)$ and $\phi_{1-}(z)$ have different behaviors, so the only way the two expressions can be equal is if

$$\int_{-\infty}^{\infty} dz'\phi_{2+}(z')q(0, z') = 0,$$

$$\int_{-\infty}^{\infty} dz'\phi_{2-}(z')q(0, z') = 0. \qquad \text{(A5)}$$

Now $\phi_{2+}(z)$ and $\phi_{2-}(z)$ are also linearly independent (since they have different behaviors as $z \to \infty$), so any homogeneous solution $\phi(z)$ can be written as a linear combination of $\phi_{2+}(z)$ and $\phi_{2-}(z)$. Hence a necessary condition for a steady state is

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dz\phi(z)q(x, z) = 0, \qquad \text{(A6)}$$

where $q(0, z)$ has been expressed as $\int_{-\infty}^{\infty} q(x, z)dx$. In the case that N and U are constant, $\phi(z)$ can be chosen as $\exp(\pm i\mu z)$, giving the condition (13).

So far, we have shown that a finite steady state displacement is possible at large positive z only if (A6) is satisfied. Now we show (A6) also guarantees the displacement is finite at other z's. At any z,

$$\lim_{k\to0^+} w(k, z) = c_+\phi_{1+}(z) \int_{-\infty}^{z} dz'\phi_{2+}(z')q(0, z') + c_+\phi_{2+}(z) \int_{z}^{\infty} dz'\phi_{1+}(z')q(0, z'), \qquad \text{(A7)}$$

and similarly for

$$\lim_{k\to 0^-} w(k, z).$$

Except at a critical level where $\phi_{1\pm}$ and $\phi_{2\pm}$ may diverge, these integrals are just pieces of integrals of the form (A6), so they do not diverge. Furthermore,

$$\lim_{k\to 0^+} w(k, z), \quad \lim_{k\to 0^-} w(k, z)$$

are both solutions of the same homogeneous Eq. (A1), so they can differ by only a homogeneous solution of (A1). The matching as $z \to \infty$ guarantees that this homogeneous solution is zero, so the two limits agree. Thus $w(0, z) = \eta(\infty, z)/U$ and hence the displacement as $x \to \infty$ is well defined and finite except possibly at critical levels. Thus, if (A6) holds, then there will indeed be a steady state.

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