Scale Analysis of Deep and Shallow Convection in the Atmosphere[1]

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ABSTRACT

The approximate equations of motion derived by Batchelor in 1953 are derived by a formal scale analysis, with the assumption that the percentage range in potential temperature is small and that the time scale is set by the Brunt-Väisälä frequency. Acoustic waves are then absent. If the vertical scale is small compared to the depth of an adiabatic atmosphere, the system reduces to the (non-viscous) Boussinesq equations. The computation of the saturation vapor pressure for *deep* convection is complicated by the important effect of the dynamic pressure on the temperature. For *shallow* convection this effect is not important, and a simple set of reversible equations is derived.

1. Introduction

In a recent paper on the behavior of convective phenomena in the atmosphere, Charney and Ogura (1960) have used a simplified form of the fundamental hydrodynamic equations for a perfect gas. For convenience and for reasons to be developed later, we will refer to this simplified set of equations as the *anelastic* equations. These anelastic equations are identical with a set of equations derived by Batchelor (1953) on what seems to be the simple assumption that the distributions of pressure and of density are always close to the distributions of pressure and density in an adiabatically stratified atmosphere. Charney and Ogura, on the other hand, were interested in the "elimination" of sound waves from the hydrodynamic equations, since sound waves require that a very small time increment be used in a numerical finite-difference integration. From this viewpoint, it is clear that an assumption about the *time scale* must be made in deriving the anelastic equations. In this paper we will present a more systematic scale analysis than was done by either Batchelor or by Charney and Ogura, and show that both assumptions are in fact necessary.

2. Basic equations and assumptions

In our analysis, molecular effects such as viscosity and conduction will be omitted. We will consider motions limited vertically by two parallel fixed boundaries separated by the vertical distance d. This distance will be used as a length unit and the symbol τ will represent an as yet arbitrary time unit. These units will be used to scale the velocities and the space and time coordinates. Instead of using the pressure p as a variable in the equations, it will prove more convenient to use the non-dimensional variable π:

$$\pi=(p/P)^{\kappa}, \tag{1}$$

where P is a reference pressure, and κ is the ratio $R/c_p=(c_p-c_v)/c_p$. In place of the density (ρ) the potential temperature will be used. The symbol θ will represent this in *non-dimensional form*, with the symbol Θ representing a constant mean value of the actual potential temperature. Thus,

$$T=\Theta\cdot\pi\theta, \tag{2}$$

where T is the absolute temperature, and both T and Θ are given in degrees Kelvin. The potential temperature is related to the specific entropy Φ by the relation

$$\begin{aligned}\Phi&=\text{constant}+c_p\ln(\Theta\theta)\\&=\text{constant}+c_p\ln\theta,\end{aligned} \tag{3}$$

while T, ρ, and p are related by the usual equation of state:

$$p=\rho RT. \tag{4}$$

The basic equations can then be written in non-dimensional form as follows:

$$\left(\frac{d^2}{\tau^2c_p\Theta}\right)\frac{d\mathbf{v}}{dt}=-\theta\nabla\pi, \tag{5}$$

$$\left(\frac{d^2}{\tau^2c_p\Theta}\right)\frac{dw}{dt}=-\theta\frac{\partial\pi}{\partial z}-\left(\frac{gd}{c_p\Theta}\right), \tag{6}$$

$$\frac{d}{dt}\left[\ln\theta+\left(1-\frac{1}{\kappa}\right)\ln\pi\right]=\nabla\cdot\mathbf{v}+\frac{\partial w}{\partial z}, \tag{7}$$

$$\frac{d\theta}{dt}=0. \tag{8}$$

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All symbols except the constants d, τ, c_p, Θ, and g in these equations are non-dimensional. $\mathbf{v}$ is the horizontal velocity and ∇ is the (non-dimensional) horizontal gradient operator. In (8) we have assumed adiabatic motion. This will later be generalized to include heating of the air by the release of latent heat under certain conditions.

Other than the assumption of adiabatic motion and neglect of the irreversible molecular processes, no physical assumptions have yet been made. The first assumption we make is that θ, the non-dimensional potential temperature, is almost constant. This may be expressed as

$$\theta = 1 + O(\epsilon). \tag{9}$$

The small number ϵ thus represents the percentage variation of potential temperature in the region.

$$\epsilon = \Delta\theta'/\Theta = \Delta\Phi/c_p. \tag{10}$$

Our second assumption will be a choice of the time scale, τ. For this we refer to the theory of small oscillations in a resting, isothermal atmosphere (Bjerknes *et al.*, 1933; Eliassen and Kleinschmidt, 1956; Eckart, 1960). This theory shows that two types of wave motion can exist under these circumstances, *acoustic* waves and *gravity* waves. The acoustic waves in general have high frequencies, while the gravity waves have low frequencies.[2] The frequency separating these two classes of wave motion is the well known Brunt-Väisälä frequency, N:

$$N^2 = g\partial \ln\theta'/\partial z'. \tag{11}$$

(Everything in this equation is dimensional.) In our analysis of deep convection we will assume that we are interested only in motions whose *time scale is similar to that of gravity waves*, and attempt to eliminate the high-frequency acoustic waves by an appropriate choice of the time unit, τ. The correct choice is clearly

$$\tau \sim N^{-1} \sim (d/g\epsilon)^{\frac{1}{2}}. \tag{12}$$

For $d \sim 10$ km, and $\epsilon \sim 0.1$, τ is about 100 sec. If our derivation of a set of approximate equations is correct, our final equations (when linearized) should only contain oscillations whose frequency is less than N. This is verified in section 4.

It will be convenient to introduce the symbol β for the non-dimensional ratio appearing in (6):

$$\beta = gd/c_p\Theta = d/H. \tag{13}$$

$H = c_p\Theta/g$ is the depth of an isentropic atmosphere with a uniform potential temperature Θ and a value of π equal to 1 at $z=0$. H is about 30 km for values of Θ typical of the troposphere. (9), (12), and (13) may now be introduced into (5)–(8):

$$\epsilon\beta d\mathbf{v}/dt = -\theta\nabla\pi, \tag{14}$$

$$\epsilon\beta dw/dt = -\theta\partial\pi/\partial z - \beta, \tag{15}$$

$$\frac{d}{dt}\left[\ln\theta + \left(1 - \frac{1}{\kappa}\right)\ln\pi\right] = \nabla\cdot\mathbf{v} + \frac{\partial w}{\partial z}, \tag{16}$$

$$d\theta/dt = 0. \tag{17}$$

3. The anelastic equations

The basic approximation we will now apply is that ϵ is a small quantity, while all other variables, parameters, and differentiations are of order unity. We expand all dependent variables as a power series in ϵ:

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + \cdots \\ w &= w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots \\ \pi &= \pi_0 + \epsilon\pi_1 + \epsilon^2\pi_2 + \cdots \\ \theta &= 1 + \epsilon\theta_1 + \epsilon^2\theta_2 + \cdots. \end{aligned} \tag{18}$$

The special first term for θ is taken from (9). Substituting these expressions into (14) and (15) and equating terms of zero order in ϵ, we get

$$\nabla\pi_0 = 0. \tag{19}$$

$$\partial\pi_0/\partial z = -\beta. \tag{20}$$

Together these show that π_0 must be of the form

$$\pi_0(z,t) = \pi_0(0,t) - \beta z. \tag{21}$$

This is the distribution of π in a hydrostatic atmosphere of uniform potential temperature equal to Θ and a variable surface pressure given by $\pi_0(0,t)$. The density in such an atmosphere is given by

$$\rho_0 = (P/R\Theta)\pi_0^{(1/\kappa)-1}. \tag{22}$$

The zero-order terms from the continuity Eq (16) may now be collected, and with the help of (21) and (22) can be condensed into the equation

$$\frac{\partial\rho_0}{\partial t} + \nabla\cdot\rho_0\mathbf{v}_0 + \frac{\partial\rho_0 w_0}{\partial z} = 0. \tag{23}$$

If we now require that w_0 vanish at $z=0$ and $z=1$, and that the horizontal average of $\nabla\cdot\rho_0\mathbf{v}_0$ also vanish (e.g., by requiring the normal component of $\mathbf{v}_0$ to vanish at fixed lateral boundaries) it can be readily shown that $\partial\rho_0/\partial t$ must vanish in (23) and that $\pi_0(0,t)$ in (21) must be a constant. This constant can be conveniently set equal to 1, by selecting a fixed value of p_0 at $z=0$ and using this for the reference pressure P. $\beta<1$ then insures that ρ_0 and π_0 are everywhere positive.

[2] The "Lamb waves" are an exception to this. They represent a degenerate type of acoustic wave with no vertical motion. They propagate horizontally with a constant velocity, so that their frequency is proportional to the horizontal wave number. The perturbation analysis in section 4 will show that these waves are not present in our modified system of equations.

The continuity equation is therefore

$$\nabla \cdot \rho_0 \mathbf{v}_0 + \partial \rho_0 w_0 / \partial z = 0, \qquad (24)$$

with ρ_0 determined by (22) and the relation

$$\pi_0 = 1 - \beta z. \qquad (25)$$

Introducing (25) into (2) we find that

$$T_0 = \Theta \pi_0 = \Theta - (g/c_p) z', \qquad (26)$$

with z' in dimensional units.

The equations necessary to complete the anelastic system are obtained by collecting the first-order terms in ϵ from (14), (15), and (17).

$$\beta d\mathbf{v}_0/dt = -\nabla \pi_1, \qquad (27)$$

$$\beta d w_0/dt = -\partial \pi_1/\partial z + \beta \theta_1, \qquad (28)$$

$$d\theta_1/dt = 0. \qquad (29)$$

To avoid excess subscripts in these and other equations, the individual derivative d/dt is now understood to be evaluated with $\mathbf{v}_0$ and w_0:

$$d/dt = \partial/\partial t + \mathbf{v}_0 \cdot \nabla + w_0 \partial/\partial z.$$

The anelastic system then consists of (24) together with (27)–(29). The relation of T_1 to θ_1 and π_1 is obtained by expanding (2):

$$(T_1/T_0) = \theta_1 + (\pi_1/\pi_0). \qquad (30)$$

This relation will be important in the discussion of the saturation vapor pressure.

The following energy conservation equation is satisfied by this system:

$$\frac{\partial}{\partial t} \int \rho_0 [\tfrac{1}{2}(\mathbf{v}_0^2 + w_0^2) - z\theta_1] dV = 0. \qquad (31)$$

The only restriction on θ_1 is that it is not greater than 1 in magnitude. In particular, the horizontal average of θ_1 need not vanish. (31) then shows that for adiabatic motion, the kinetic energy will increase only if θ_1 decreases at small values of z and increases at larger values of z, i.e., if the average static stability is increased.

If dimensional variables are introduced into (31), the term $-z\theta_1$ becomes $-gz'(\epsilon\theta_1)$, where z' is the actual height. On the other hand, if the usual expression $\rho(c_v T + gz')$ for the internal and potential energy per unit volume is expanded in ϵ, we obtain the following expression

$$\rho(c_v T + gz') = \rho_0(c_v T_0 + gz') + \epsilon \rho_0 [c_v \Theta (p_1/p_0) - gz'\theta_1] + \cdots.$$

The term (p_1/p_0) appearing here has evidently been eliminated in (31) by our choice of τ.

Eckart and Ferris (1956) have considered the "external energy density," E, of small *linearized* perturbations on a resting stratified atmosphere. E consists of kinetic energy, "thermobaric" energy, and "elastic" energy. The elastic energy is proportional to the square of the pressure perturbation. If the set of Eq (24)–(29) are subjected to the same linearization, they yield the same form for E except that the elastic energy is missing. For this reason, it seems appropriate to use the adjective "anelastic" to describe them.[3] "Sound-proof" might also be appropriate, as will be demonstrated in the next section.

Another important point concerning the anelastic system is that the variables $\mathbf{v}_0$, w_0, π_1, and θ_1 are not completely independent; the distribution of π_1 must always be such that the accelerations in (27) and (28) continue to satisfy the continuity Eq (24). This is pointed out by Batchelor (1953) and is also encountered in the theory of incompressible flows. The net result is that π_1 must be determined by the solution of an elliptic equation:

$$\nabla \cdot \rho_0 \nabla \pi_1 + \frac{\partial}{\partial z}\left(\rho_0 \frac{\partial \pi_1}{\partial z}\right) = \beta \frac{\partial \rho_0 \theta_1}{\partial z} - \beta \nabla \cdot \left[\rho_0 \left(\mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + w_0 \frac{\partial \mathbf{v}_0}{\partial z}\right)\right] - \beta \frac{\partial}{\partial z}\left[\rho_0 \left(\mathbf{v}_0 \cdot \nabla w_0 + w_0 \frac{\partial w_0}{\partial z}\right)\right]. \qquad (32)$$

Boundary conditions for π_1 are given by the vanishing of $\nabla \pi_1$ at lateral boundaries and of $(\partial \pi_1/\partial z - \beta \theta_1)$ at the top and bottom boundaries. These boundary conditions determine π_1 except for an arbitrary constant.

4. Perturbation solutions of the anelastic equations

We consider small perturbations—$\mathbf{v}_0'$, w_0', π_1', and θ_1', superimposed on a *basic* state given by $\bar{\theta}_1$:

$$\bar{\theta}_1 = \sigma z, \quad \sigma = \text{constant}.$$

According to the scale analysis, the size of σ is at most ± 1.

Let the *disturbance* quantities be of the following form:

$$\mathbf{v}_0' = [U(z), V(z)] \cdot \psi$$
$$w_0' = W(z) \cdot \psi$$
$$\pi_1' = \beta \Pi(z) \cdot \psi$$
$$\theta_1' = S(z) \cdot \psi.$$

Here $\psi = \exp i(kx + ly + \nu t)$. The linearized perturbation

[3] Suggested by J. Charney.

equations may then be written as

$$i\nu U=-ik\Pi,$$
$$i\nu V=-il\Pi,$$
$$i\nu W=-d\Pi/dz+S,$$
$$i\nu S+\sigma W=0$$
$$i\rho_0(kU+lV)+d(\rho_0 W)/dz=0.$$

The following differential equation for W is readily obtained:[4]

$$\frac{d}{dz}\left[\frac{1}{\rho_0}\frac{d\rho_0 W}{dz}\right]+\lambda^2 W=0, \tag{33}$$

where $\lambda^2=(k^2+l^2)(\sigma-\nu^2)\nu^{-2}$. The same equation is also satisfied by W^*, the conjugate of W.

We require that W and W^* satisfy the boundary conditions

$$W \text{ and } W^*=0 \quad \text{at} \quad z=0 \quad \text{and} \quad z=1.$$

Multiplication of (33) by $\rho_0 W^*$, followed by integration from $z=0$ to 1, then produces the following equality:

$$\int_0^1 \frac{1}{\rho_0}\left|\frac{d\rho_0 W}{dz}\right|^2 dz=\lambda^2\int_0^1 \rho_0|W|^2 dz.$$

Both integrands are positive definite, showing that λ^2 is positive. Solving for ν^2, we obtain

$$\nu^2=\sigma(k^2+l^2)/(k^2+l^2+\lambda^2). \tag{34}$$

Since σ is at most equal to 1 in magnitude, the positive nature of λ^2 ensures that the magnitude of ν is less than or equal to 1, as was implied by our choice of time scale (12).

5. Comparison with the analysis of Batchelor

Batchelor's form of the anelastic equations (Batchelor, 1953) is given by his Eq (22), (23) and (24). They are formally identical with our Eq (24)–(29) if the following relations hold between our variables and those of Batchelor:

$$\begin{aligned} p_0&=p_a,\\ \rho_0&=\rho_a,\\ \epsilon p_1&=p-p_a,\\ \epsilon\theta_1&=(\rho^*-\rho_a^*)/\rho_a^*. \end{aligned} \tag{35}$$

These relations can be readily verified by first rewriting our system in dimensional units and then expanding the variables π_1 and θ_1 in terms of p and ρ by means of (1), (2), and (4).

Our analysis has definitely required two assumptions, namely the restriction of $\epsilon=\Delta\theta'/\Theta$ to small values, and the selection of the time scale τ as that given by the reciprocal of the Brunt-Väisälä frequency. It might appear from Batchelor's derivation of his Eq (22)–(24) that he had made only one assumption—the closeness of p and ρ to p_a and ρ_a—and that no assumption about the time scale was necessary in his development. It can be shown, however, that there is an implicit assumption about the time scale present in his analysis.

Batchelor's time unit is equal to the quotient of a typical length L, divided by a typical velocity, U. In the derivation of his continuity Eq (23),

$$\nabla\cdot\mathbf{v}+\frac{\partial w}{\partial z}=-\frac{1}{\rho_a}\frac{d\rho_a}{dt}=-\frac{w}{\rho_a}\frac{\partial\rho_a}{\partial z},$$

from the original equation

$$\nabla\cdot\mathbf{v}+\partial w/\partial z=-\rho^{-1}d\rho/dt,$$

it is necessary to expand the right side of the latter equation in the form

$$-\frac{1}{\rho_a}\frac{d\rho_a}{dt}-\left(1+\frac{\rho-\rho_a}{\rho_a}\right)^{-1}\frac{d}{dt}\left(\frac{\rho-\rho_a}{\rho_a}\right). \tag{36}$$

The essential part of Batchelor's argument might seem to be simply that $(\rho-\rho_a)/\rho_a$ is small and that therefore the last term in (36) can be neglected. But this term can really be neglected only if

$$\frac{\partial}{\partial t}\left(\frac{\rho-\rho_a}{\rho_a}\right)\sim U\frac{\partial}{\partial x}\left(\frac{\rho-\rho_a}{\rho_a}\right)\sim\frac{U}{L}\left(\frac{\rho-\rho_a}{\rho_a}\right), \tag{37}$$

is small. At this point it is necessary to interpret Batchelor's general assumption about small Mach number as signifying not only the fact that the velocities are smaller than the speed of sound, but also the important restriction that there are no frequencies larger than U/L.

This restriction is equivalent to the implicit assumption we have made that the velocities become of order unity when multiplied by τ/d. Thus we can write

$$\frac{U}{L}\sim\frac{U}{d}=\frac{u}{\tau}=uN, \tag{38}$$

showing that an assumption that frequencies are limited by U/L is equivalent to the two assumptions that frequencies are limited by the Brunt-Väisälä frequency N and that $u\sim 1$. If a Richardson number is defined by $(Nd/U)^2$, our scaling assumptions are equivalent to assuming a Richardson number of order 1 in magnitude (rather than 0.1 or 10, say).

6. Shallow convection

So far we have treated β as a parameter which, while it is not greater than 1, is also not small. In this section

[4] The special case $W\equiv 0$ corresponds only to steady horizontal rotational motion, with $\nu=0$. Lamb waves are therefore absent.

we will consider the case of small β:

$$d \ll H = c_p \Theta / g.$$

Considering still the case of dry-adiabatic motion, we take the anelastic system of section 3, where the variables are π_0, ρ_0, $\mathbf{v}_0$, w_0, π_1, and θ_1. We expand each of these as a series in β:

$$\begin{aligned} \pi_0 &= \pi_{00} + \beta \pi_{01} + \cdots, \\ \pi_1 &= \pi_{10} + \beta \pi_{11} + \cdots, \end{aligned} \tag{39}$$

and collect terms of equal powers of β. [Note that (9) implies that θ_{01} is zero.]

We first find that

$$\begin{aligned} &\pi_{00} = 1, \quad \pi_{01} = -z \\ &\rho_{00} = P(R\Theta)^{-1} = \text{constant}, \\ &\pi_{10} = \text{constant}. \end{aligned} \tag{40}$$

(It is convenient to take π_{10} equal to zero.) The following system of prediction equations is then obtained

$$\nabla \cdot \mathbf{v}_{00} + \partial w_{00} / \partial z = 0, \tag{41}$$

$$d\mathbf{v}_{00}/dt = -\nabla \pi_{11}, \tag{42}$$

$$dw_{00}/dt = -\partial \pi_{11}/\partial z + \theta_{10}, \tag{43}$$

$$d\theta_{10}/dt = 0. \tag{44}$$

The operator d/dt is now equal to

$$\partial/\partial t + \mathbf{v}_{00} \cdot \nabla + w_{00} \partial / \partial z.$$

We note that these equations already possess the general form of the incompressible Boussinesq system, which has found extensive use in analyzing small-scale convection in liquids. [In many of those problems, however, the effects of viscosity and heat conduction must be retained. See Spiegel and Veronis (1960).] The interpretation of π_{11} and θ_{10} is more readily seen if (1), (2), (4) and (9) are expanded in terms of ϵ and β. The relevant results are

$$\begin{aligned} T_{00} &= \Theta, \\ T_{01} &= -\Theta z, \\ T_{10} &= \Theta \theta_{10}, \\ \pi_{11} &= (\tau^2/d^2)(\epsilon\beta)(p_{11}/\rho_{00}). \end{aligned} \tag{45}$$

Comparing the last of these with (42) and (43), and noting that the factor (τ^2/d^2) can be used to put $\mathbf{v}_{00}$, w_{00}, t, ∇ and $\partial/\partial z$ back into dimensional form, we see that π_{11} represents the deviation of the *pressure* from that of an adiabatic atmosphere $\pi_{00}+\beta\pi_{01}$. The temperature distribution to the first order in ϵ and β can be expressed as

$$T_{00} + \beta T_{01} + \epsilon T_{10} = \Theta - (g/c_p) z' + \epsilon \Theta \theta_{10}, \tag{46}$$

where z' is in dimensional units. θ_{10} therefore represents the deviation of the *temperature* from that in an adiabatic atmosphere. This is to be contrasted with the case of deep convection, where (30) shows that θ_1 represents the deviation of *potential temperature.*

The partial analysis of the equations given by Jeffreys (1930) is equivalent to the former interpretation of θ. Batchelor (1953) has also derived the system (41)–(44) as representing the most general system in which the Richardson number was the only physical similarity parameter. Here we point out that according to (46), his variable $(\rho^* - \rho_a^*)/\rho_a^*$ is (for small depth) even simpler than his Eq (31) indicates.

7. Release of latent heat

The specification of the saturation vapor pressure e_s (a function of T) is important for both reversible and non-reversible descriptions of the condensation process. We consider T expanded in the form $T_0 + \delta T_1 + \delta^2 T_2 +$ etc., where δ is some small parameter, and examine the error in e_s resulting from truncation of this expansion for T. To do this we use the Clausius-Clapeyron equation, assuming water vapor to be an ideal gas and liquid water to be incompressible:

$$\frac{1}{e_s}\frac{de_s}{dT} = \frac{L}{R_v T^2}$$

($L \sim 2.5 \times 10^6$ kj ton^{-1} is the latent heat, and $R_v = 462$ kj ton^{-1} deg^{-1} is the gas constant for water vapor). We obtain for e_s the formula

$$e_s = e_s(T_0) \exp\left[\frac{\delta L_0}{R_v T_0}\left(\frac{T_1}{T_0}\right)\right] \times \left\{1 + \left(\frac{\delta L_0}{R_v T_0}\right)\delta\left[\frac{T_2}{T_0} - \left(\frac{T_1}{T_0}\right)^2\right] + \cdots\right\}. \tag{47}$$

The first order term in δ has been deliberately retained in the exponential because the size of the non-dimensional ratio $L_0/R_v T_0$ must be considered in making this expansion. For typical atmospheric temperatures this ratio is about 19.

If δ is identified with the ϵ of the anelastic system, it has a value of about 0.1 for deep moist convection ($d \sim 10$ km). Under these conditions, $(\delta L_0/R_v T_0)$ is about 2, so the exponential form must be kept. Considering then the case where the series for e_s is truncated at T_1, we have

$$e_s = e_s(T_0) \exp\left[\frac{\epsilon L_0}{R_v T_0}\left(\frac{T_1}{T_0}\right)\right]. \tag{48}$$

Sample calculations with this formula show that the percentage error in e_s is of the order of ϵ, i.e., 1/10.

However, a more serious complication in the case of deep convection is introduced by the fact that according to (30), T_1 can be computed from θ_1 only if π_1 is known. π_1 is given by the elliptic Eq (32), whose solution presumes that θ_1 is already known. But θ_1 is now no longer given simply by the adiabatic Eq (29), but can be computed only if the rate of release of latent heat is known. This in turn requires a knowledge of e_s, resulting in a highly implicit relation between the thermodynamic variables.

The writers have attempted to simplify the treatment of deep moist convection by considering an alternate expansion about a saturated moist-adiabatic atmosphere in place of the anelastic expansion about a dry-adiabatic atmosphere. The advantage of this is that T_0 would now correspond to a moist-adiabatic lapse rate and δ would be of the order 0.01 (i.e., corresponding to about 3 deg deviation in temperature) for most cases of interest. The very simple approximation $e_s \sim e_s(T_0)$ might then be employed with some justification, eliminating the need to know T_1. However, we have been unable to develop a satisfactory expansion of this type, primarily because of the complications introduced by precipitation, freezing, and the heat content of liquid water (Saunders, 1957). These effects are negligible if a 10 per cent error is made in the hydrodynamic equations, but are not negligible if an error of only 1 per cent is made in the hydrodynamics.

We will therefore limit our further presentation to the case of small β ($d \sim 3$ km, say), where (45) shows that the first-order temperature does not involve the dynamic pressure π_{11}.

Considering for simplicity a reversible process, it is easily demonstrated that the (variable) entropy may be approximated to order ϵ by the expression

$$\phi = \theta_{10} + B r_v, \tag{49}$$

where $B = L(c_p T_{00} \epsilon)^{-1}$ and r_v is the mixing ratio of water vapor to dry air (the density of which is ρ_{00} = constant). (41)–(43) remain as the hydrodynamic equations,[5] but (44) is replaced by the statement

$$d\phi/dt = 0. \tag{50}$$

This equation predicts the variation of ϕ in space and time, allowing θ_{10} to be determined from (49) if r_v is specified.

We define r as the combined mixing ratio of total water (vapor and liquid) to dry air. For a reversible process

$$dr/dt = 0; \quad r = r_l + r_v. \tag{51}$$

The saturation mixing ratio r_{vs} is defined using (48) for e_s:

$$\begin{aligned} r_{vs} &= \frac{R}{R_v P} e_s(T_0) \exp(A\theta_{10}) \\ &= Z(z) \exp(A\theta_{10}). \end{aligned} \tag{52}$$

Here the constant A is equal to $\epsilon L / R_v \Theta$, and the symbolic T_0 appearing in $e_s(T_0)$ in (48) has been interpreted as representing $T_{00} + \beta T_{01}$. $Z(z)$ is a monotonic decreasing function of z.

[5] The effect of the drag of liquid water may be added to (43) if desired.

Saturated and unsaturated conditions are defined by

$$\text{Saturated:} \quad r > r_{vs}; \quad r_v = r_{vs}, \quad r_l = r - r_{vs}.$$
$$\text{Unsaturated:} \quad r < r_{vs}; \quad r_v = r, \quad r_l = 0.$$

Considering ϕ and r to be specified by the conservation Eq (50) and (51), it is then easily shown that a particle is saturated if and only if

$$Z(z) < r \exp[A(Br - \phi)]. \tag{53}$$

This result shows that in this greatly idealized condensation model each particle has a characteristic condensation level z_c, which is specified as a function of r and ϕ for that particle by the implicit relation $Z(z_c) = r \exp[A(Br-\phi)]$. For $z < z_c$, the particle is unsaturated, and we have

$$\theta_{10} = \phi - Br = \text{constant}. \tag{54}$$

When the particle is at a higher elevation, θ_{10} is given by the implicit relation

$$\theta_{10} + BZ(z) \exp(A\theta_{10}) = \phi, \tag{55}$$

and increases monotonically for that particle with increasing z. θ_{10} is therefore a unique function $M(z)$ for each particle, the function M varying from particle to particle. We then define $N(z)$ for each particle by the integral relation

$$N(z) = \int_z^1 M dz. \tag{56}$$

The energy equation for shallow moist reversible convection may then be written

$$\frac{\partial}{\partial t} \int [\tfrac{1}{2}(\mathrm{v}_{00}^2 + w_{00}^2) + N] dV = 0, \tag{57}$$

showing that energy is conserved in this system. A schematic picture of the variation with z of M (i.e., θ_{10}) and N is shown in Fig. 1.

8. Concluding remarks

In this section we will draw attention to several general implications of our analysis which might otherwise be overlooked in the details of the preceding sections.

Many investigations have been made of the ascent of "bubbles" and other simplified cloud models. [See, for example, Mason and Emig (1961).] While our analysis of the equations has been based on a very simple condensation process and laminar motion, it indicates how the accuracy with which the hydrodynamics is treated has an effect on the accuracy with which the thermodynamic relations should be expressed. The analysis further suggests that deep con-

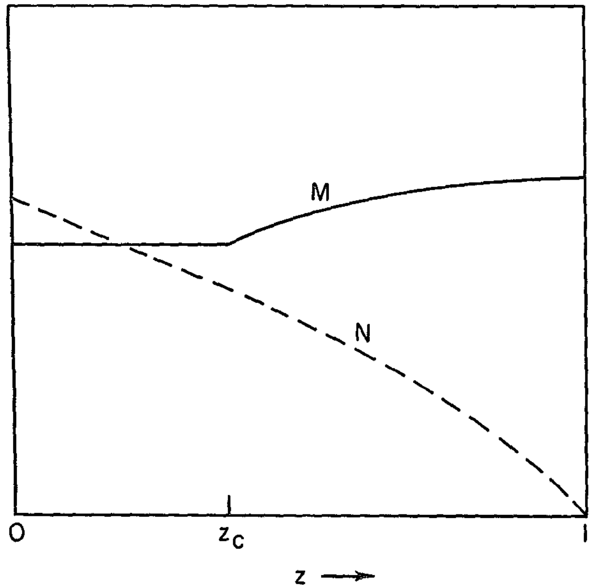

FIG. 1. Schematic indication of the dependence on z of the potential temperature M and potential energy function N for a particle whose condensation level is z_c.

vection may differ qualitatively from shallow convection; in the latter case the saturation vapor pressure is not appreciably affected by the dynamic pressure, while in deep convection the dynamic pressure may be large enough to have a serious effect on the saturation vapor pressure. (The conventional practice of using a hydrostatic pressure field to compute e_s from the local value of the entropy is therefore justified for shallow convection, but would seem to be suspect for deep convection.)

Several authors have recently used the Boussinesq system as a basis for numerical computation experiments (Malkus and Witt, 1959; Fisher, 1961), and stated that this involves an implicit assumption of hydrostatic balance. It is clear from our analysis (and from that of Batchelor) that this is a misinterpretation; the only limit on the acceleration is that set by the depth and the Brunt-Väisälä frequency.

Finally, we point out that non-adiabatic heating (Q) by processes other than latent heat may be introduced into (29) or (44) without violating the scale analysis if $Q\tau(\epsilon c_p T)^{-1}$ is of order unity.

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