Brutally simple 1D example of anelastic pressure equation + its solution

\[ \frac{d^2 P}{dx^2} = F \]

discretize. Drop primes.

\[ \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = F_i \]

rewrite:

\[ \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} = F_i \Delta x^2 \quad (*) \]

domain with \( N \) real points + 2 boundary points \( (i=0, i=N+1) \)

\[ \text{Need to specify how } T_i \text{ at boundaries is handled} \]

-2 types of BCs:
- Dirichlet BC - \( T_i \) at boundary is specified or known
- Neumann BC - gradient of \( T_i \) is specified or known
- For this example, presume
\( \pi_0 = \pi_{N+1} = \text{specified} \)

- For the unknown non-boundary point, we can create a matrix

\[
\begin{bmatrix}
-2 & 1 & 0 & \ldots & 0 \\
1 & -2 & 1 & \ldots & 0 \\
0 & 1 & -2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & -2
\end{bmatrix}
\begin{bmatrix}
\pi_1 \\
\pi_2 \\
\pi_3 \\
\vdots \\
\pi_{N-2} \\
\pi_{N-1} \\
\pi_N
\end{bmatrix}
= \begin{bmatrix}
F_1 - \pi_0 \Delta x^2 \\
F_2 \\
F_3 \\
\vdots \\
F_{N-1} \\
F_N - \pi_{N+1} \Delta x^2
\end{bmatrix}
\]

\( N \times N \quad N \times 1 \quad N \times 1 \)

- Note regular structure of \( N \times N \) matrix

- At any interior point not involving boundaries, \((x) \rightarrow \)

\[
\pi_{i+1} - 2\pi_i + \pi_{i-1} = \Delta x^2 \cdot F_i
\]

- Coefs \( 1 \quad -2 \quad 1 \)

- No dependence on \( i-2 \) or \( i+2 \). These coeffs are 0
at \( t = 1 \) (closest interior point to left boundary)

\[
\Pi_0 - 2\Pi_1 + \Pi_2 = \Delta x^2 F_1
\]

but here \( \Pi_0 \) specified
move to RHS

\[
= -2\Pi_1 + \Pi_2 = \Delta x^2 \left[ F_1 - \frac{\Pi_0}{2\Delta x^2} \right]
\]

at \( t = N \), \( \Pi_{N+1} \) is known

\[
\Pi_{N+1} - 2\Pi_N = \Delta x^2 \left[ F_N - \frac{\Pi_{N+1}}{\Delta x^2} \right]
\]

Equation (N+1) has form

\[
A \Pi = F
\]

\[
\begin{bmatrix}
\Pi_N & \Pi_{N+1} & \Pi_{N+1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
A & \Pi_N & \Pi_{N+1}
\end{bmatrix}
\]

\[
\rightarrow A \text{ known matrix of coefficients from Equation}
\]

\[
\rightarrow F \text{ known - calculated at time } N \text{; includes BCs}
\]

\[
\Pi \text{ vector of unknown pressure perturbations}
\]

Need to invert \( \Pi = A^{-1} F \)
Many Ways to Solve A

- Gaussian elimination is obvious candidate but very slow in practice

- Iterative and direct methods exist - not so nearly details (see M76)

Teeny problem \( N=3 \), 3 interior pts

\[
\begin{bmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2
\end{bmatrix}
\begin{bmatrix}
\pi_1 \\
\pi_2 \\
\pi_3
\end{bmatrix}
= \begin{bmatrix}
3 \\
1 \\
2
\end{bmatrix}
\]

Assumed: \( \Delta x^2 = 1 \)

\( \pi_0 = \pi_{N+1} = 0 \)
\( F_1 = 3 \quad F_2 = 1 \quad F_3 = 2 \)

- Gaussian elimination yields

\( \pi_1 = -\frac{13}{4} \quad \pi_2 = -\frac{3}{2} \quad \pi_3 = \frac{11}{4} \)

- Cannot solve any \( \pi_i \) w/o also solving 2D neighbors

- Ultimately, all points depend on the boundaries!

- BCS can be a real problem!

- Should not need boundary cond for \( \pi_1 \) on a staggered grid anyway!