Brutally simple 1D example of anelastic pressure equation + its solution

$$\frac{\partial^2 \pi'}{\partial x^2} = F$$

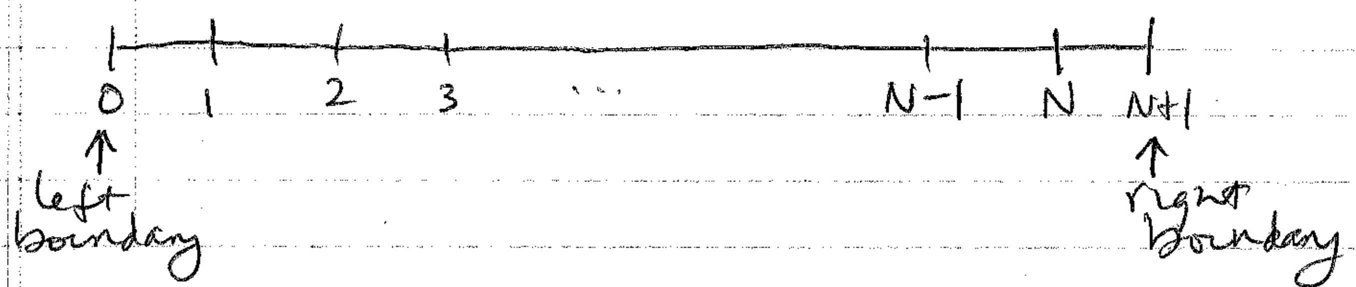
- discretize. Drop primes

$$\frac{\pi_{i+1} - 2\pi_i + \pi_{i-1}}{\Delta x^2} = F_i$$

- rewrite

$$\pi_{i+1} - 2\pi_i + \pi_{i-1} = F_i \Delta x^2 \qquad (*)$$

- domain w/ N real points + 2 boundary points ($i=0$, $i=N+1$)



- Need to specify how π @ boundaries is handled
  - 2 types of BCs
    - Dirichlet BC – π @ boundary is specified or known
    - Neumann BC – gradient of π is specified or known

- for this example, presume
  $\pi_0 = \pi_{N+1}$ = specified

- for the unknown non-boundary points, we can create a matrix

$$(**) \quad \underbrace{\begin{bmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{N \times N} \underbrace{\begin{bmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{N-1} \\ \pi_N \end{bmatrix}}_{N \times 1} = \Delta x^2 \underbrace{\begin{bmatrix} F_1 - \pi_0/\Delta x^2 \\ F_2 \\ \vdots \\ F_{N-1} \\ F_N - \frac{\pi_{N+1}}{\Delta x^2} \end{bmatrix}}_{N \times 1}$$

· note regular structure of $N \times N$ matrix

· at any interior point not involving boundaries, $(*) \rightarrow$

$$\pi_{i+1} - 2\pi_i + \pi_{i-1} = \Delta x^2 F_i$$

coefs	1	−2	1	RHS

- no dependence on $i-2$ or $i+2$. Those coefs are $\emptyset$

· at $i=1$ (closest interior point to left boundary)

$$\pi_0 - 2\pi_1 + \pi_2 = \Delta x^2 F_1$$

but here π_0 specified; move to RHS

$$\therefore -2\pi_1 + \pi_2 = \Delta x^2\left[F_1 - \frac{\pi_0}{\Delta x^2}\right]$$

· at $i=N$, π_{N+1} is known

$$\therefore \pi_{N-1} - 2\pi_N = \Delta x^2\left[F_N - \frac{\pi_{N+1}}{\Delta x^2}\right]$$

Equation (**) has form

$$\underset{N\times N}{\underset{\sim}{A}}\ \underset{N\times 1}{\vec{\pi}} = \underset{N\times 1}{\vec{F}}$$

→ $\underset{\sim}{A}$ KNOWN matrix of coefficients from equation

→ $\vec{F}$ KNOWN – calculated at time n; includes BCs

→ $\vec{\pi}$ vector of UNKNOWN pressure perturbations

· Need to invert ⇒ $\boxed{\vec{\pi} = \underset{\sim}{A}^{-1}\vec{F}}$

· Many ways to invert $\underset{\sim}{A}$

- Gaussian elimination is obvious candidate but very slow in practice
- Iterative and direct methods exist – not sweating details (see MT6)

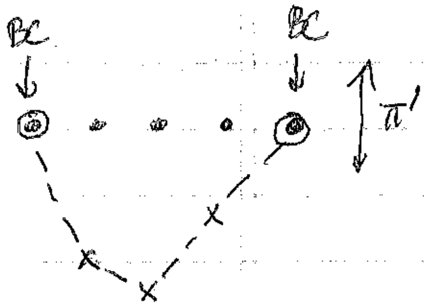
Teeny problem $N=3$, ~~one~~ 3 interior pts

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

assumed: $\Delta x^2 = 1$

$\pi_0 = \pi_{N+1} = 0$

$F_1 = 3 \quad F_2 = 1 \quad F_3 = 2$



- Gaussian elimination yields

$$\pi_1 = -\frac{13}{4} \quad \pi_2 = -\frac{7}{2} \quad \pi_3 = -\frac{11}{4}$$

- Cannot solve any π w/o also solving its neighbors

∴ Ultimately ALL POINTS DEPEND on the BOUNDARIES!

∴ BCs can be a real problem!

→ SHOULD NOT NEED boundary cond for π' ON A STAGGERED GRID anyway!

→ When Neumann BCs applied, π' has NO UNIQUE SOLUTION
- only get π' to within an UNKNOWN constant
- Advantage of π' over p'!