A Re-examination of the "Leipzig Wind Profile" Considering some Relations between Wind and Turbulence in the Frictional Layer

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Abstract

The "Leipzig Wind Profile" is a unique example of a representative wind distribution in the frictional layer which resulted from MILDNER's set of 28 pilot-balloon observations with two theodolites on October 20, 1931, near Leipzig for a stable weather situation. It is shown, that the vectors of geostrophic motion and ground drag follow from the wind profile when a scalar austausch is assumed. The re-computed vertical austausch-distribution indicates that the energy for maintaining the turbulence in a steady frictional wind profile is taken from the potential energy of the horizontal pressure field by means of the steady flow of air across the isobars.

The frictional layer is the atmospheric layer adjacent to the ground where turbulent stresses are significant. Consequently, the frictional layer is characterized by considerable deviations of the observed wind from the theoretical air motion which follows from the pressure distribution with the aid of the equations of motion when friction is neglected.

Let us assume that the wind in any given horizontal plane is a steady uniform rectilinear flow. Then, the variations of the turbulent stresses in the horizontal directions are negligible such that only the vertical variations need be considered. The equation of motion for this case is

$$\mathbf{T}' = \varrho f i\,(\mathbf{v} - \mathring{\mathbf{v}}) \tag{1}$$

when the prime denotes the partial differentiation with respect to height z (meters) and

$\mathbf{T}$ = horizontal vector of turbulent stress, dynes/cm²;
$\mathbf{v}$ = horizontal wind vector, m/sec;
$\mathring{\mathbf{v}}$ = horizontal geostrophic wind vector, m/sec;
f = Coriolis parameter = 0.0001458 sin Φ, sec^{-1};
Φ = geographic latitude, degrees;
ϱ = density of the air, g/cm³;
$i = \sqrt{-1}$.

A vector notation is used where the unit vector in the horizontal plane is $\mathbf{i}$ when $i\,\mathbf{i}$ denotes the horizontal direction perpendicular to $\mathbf{i}$. The positive $i\,\mathbf{i}$ direction is defined in fig. 1.

Eq. (1) shows that—for steady states—the internal friction ($\mathbf{T}'$) balances the pressure gradient ($\nabla p = -\varrho f i\,\mathring{\mathbf{v}}$) and the Coriolis force ($\varrho f i\,\mathbf{v}$); the physical units of the terms in eq. (1) are dynes/cm³.

Let us define the turbulent stress as proportional to the wind shear, i.e.

$$\mathbf{T} = A\,\mathbf{v}' \tag{2}$$

where A = austausch-coefficient, g/cm sec; A is invariably positive and a function of z. In general, a linear relation between two vectors—as in eq. (2)—defines a tensor. However, it will be shown below that a scalar austausch coefficient is sufficient and—owing to our present knowledge of wind distributions—necessary to describe the characteristics of turbulence in the frictional layer.

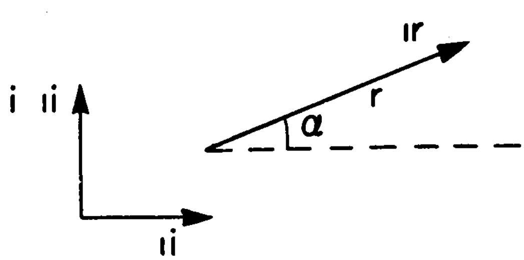

Fig. 1. The unit vectors and an arbitrary vector in a horizontal plane.

Let r and α be the absolute value and the azimuth of an arbitrary horizontal vector $\mathbf{r}$; see fig. 1. Then

$$\mathbf{r} = \mathbf{i}\, r\, e^{i\alpha} = \mathbf{i}\, r \cos\alpha + i\, \mathbf{i}\, r \sin\alpha \quad (3)$$

The differentiation of eq. (3) with respect to z yields:

$$\mathbf{r}' = \mathbf{r}\left(\frac{r'}{r} + i\,\alpha'\right) \quad (4)$$

Since the scale product of two vectors which are perpendicular to each other equals zero, we obtain from eq. (4) the following geometrical relations:

$$\mathbf{r} \cdot i\,\mathbf{r} = \mathbf{r}' \cdot i\,\mathbf{r}' = 0 \quad (5\text{ a})$$

$$\mathbf{r}' \cdot i\,\mathbf{r} = -r^2\alpha' \quad (5\text{ b})$$

Upon multiplying eqs. (1) and (2) by $i\,\mathbf{v}$ and $i\,\mathbf{v}'$, respectively, when eq. (5 a) is considered, we obtain

$$i\,\mathbf{v} \cdot \mathbf{T}' = -\varrho f \mathbf{v} \cdot (\mathbf{v} - \mathring{\mathbf{v}}) \quad (6)$$

$$i\,\mathbf{v}' \cdot \mathbf{T} = i\,\mathbf{v}' \cdot \mathbf{v}' A = 0 \quad (7)$$

thus:

$$i\,\mathbf{v} \cdot \mathbf{T}' = (i\,\mathbf{v} \cdot \mathbf{T})' = -\varrho f \mathbf{v} \cdot (\mathbf{v} - \mathring{\mathbf{v}}) = -\varrho f (v^2 - v\mathring{v} \cos\alpha). \quad (8)$$

where v and $\mathring{v}$ are the absolute values of $\mathbf{v}$ and $\mathring{\mathbf{v}}$ when α is the angle between $\mathbf{v}$ and $\mathring{\mathbf{v}}$. Let T be the absolute value of $\mathbf{T}$ and let the subscript o denote the boundary values of v, T and α at the ground level $z = 0$. The boundary conditions of the problem are

$$v_0 = 0,\ T_0 \neq 0,\ 0 < \alpha_0 < \frac{\pi}{2}; \quad (9)$$

with the aid of eqs. (5 b) and (9) it follows from eq. (8) that

$$A = \frac{1}{v^2\alpha'} \int_0^z \varrho f (v^2 - v\,\mathring{v} \cos\alpha)\, dz \quad (10)$$

Eq. (10) can be used for the direct determination of A at any level of the frictional layer provided v, α and $\mathring{v}$ are known with sufficient accuracy. Eq. (10) corresponds to the formulas of SOLBERG (see V. BJERKNES and collaborators, 1933) and FJELSTAD (1929, see also SVERDRUP, 1933).

Wind observations, which are accurate and representative enough in order to permit the application of eq. (10) are extremely scarce. MILDNER'S (1932) set of 28 pilot-balloon observations with two theodolites on October 20, 1931, near Leipzig, appears to be the only one which is reliable with regard to the assumptions of steady and uniform rectilinear flow and the elimination of random turbulent fluctuations.

By means of a graphical differentiation as based on a smoothed line connecting the endpoints of the averaged observed v-vectors in the frictional layer, MILDNER obtained with the aid of SOLBERG'S formula and the geostrophic wind from the weather maps the following A-values (see Table 1).

Table 1. MILDNER'S austausch-distribution.

z, m.......	80	125	190	240	295	405	460	510
A, g/cm sec.	125	270	310	500	246	117	70	70

For the sake of brevity let us refer to Mildner's wind observations as the "Leipzig Wind Profile."

In re-examining the "Leipzig Wind Profile" this author found that the results which follow from eq. (10) are considerably affected by relatively small variations of the azimuth of the geostrophic wind. It had to be concluded that the geostrophic wind as taken from the weather map might be too inaccurate in comparison with the value derived from the wind profile when certain geometrical relations are considered.

Let the subscripts x and y denote the horizontal vector components in the directions of $\mathbf{i}$ and $i\,\mathbf{i}$, respectively. For convenience, let us assume that $\mathring{\mathbf{v}}' = 0$ and let the direction of $\mathbf{i}$ be parallel to $\mathring{\mathbf{v}}$. The assumption $\mathring{\mathbf{v}}' = 0$ is generally a satisfactory one; it means that the horizontal density distribution is uniform or thermal winds are negligible.

Moreover, let us assume that in the frictional layer the terms $\varrho'\,\mathbf{v}$ are negligible in comparison with $\varrho\,\mathbf{v}'$. Considering the relative shallowness of the atmospheric layer in question ($0 \leq z \leq 1{,}000$ m), $\varrho' = 0$ appears to be a tolerable approximation.

There exist the two significant levels z_1 and z_2 in the frictional layer. At $z = z_1$, v_y, the wind component perpendicular to $\mathring{\mathbf{v}}$, reaches its first extreme value when at $z = z_2$, v_x, the wind component parallel to $\mathring{\mathbf{v}}$, reaches its first extreme value. Owing to the geometrical shape of the wind profile and the boundary conditions $v_{x0} = v_{y0} = 0$ at $z = 0$, these extreme values are maximum values.

Eq. (1) yields, with the aid of eqs. (9) and the above definitions and assumptions:

$$T_x = T_{0x} - \varrho f \int_0^z v_y \, dz = A \, v_x' \quad \text{(11a)}$$

$$T_y = T_{0y} + \varrho f \int_0^z (v_x - \mathring{v}) \, dz = A \, v_y' \quad \text{(11b)}$$

Since $A > 0$ at any level, when $v_x' = 0$ at $z = z_2$ and $v_y' = 0$ at $z = z_1$, we must have

$$T_{0x} = \varrho f \int_0^{z_2} v_y \, dz \qquad \text{(12 a)}$$

$$T_{0y} = - \varrho f \int_0^{z_1} (v_x - \mathring{v}) \, dz \qquad \text{(12 b)}$$

The direction of $\mathbf{T}_0$ follows from the direction of the wind near the ground. Unknown values are the correct amounts of α_0 and $\mathring{v}$, i.e. the azimuth (relative to $\mathbf{T}_0$) and the value of the geostrophic wind.

For evaluating $\mathring{\mathbf{v}}$ from the "Leipzig Wind Profile", the following method was used: A tentative α_0 is assumed; then, the observations yield v_x, v_y, z_1 and z_2; from eq. (12 a), T_{0x} can be computed which yields T_{0y} by means of the general relation:

$$T_{0y} = T_{0x} \tan \alpha_0; \qquad (13)$$

with the aid of eq. (12 b) one obtains:

$$\mathring{v} = \frac{1}{z_1} \left\{ \int_0^{z_1} v_x \, dz + \tan \alpha_0 \int_0^{z_2} v_y \, dz \right\} \quad (14)$$

Four different values of α_0 were used in re-examining the "Leipzig Wind Profile" and resulted in different values of z_1, z_2, T_0 and $\mathring{v}$ (see Table 2). All computations were carried out for successive 50 m-levels with the aid of graphical-numerical integrations.

Table 2. Characteristics of the "Leipzig Wind Profile" for different α_0.

α_0	23.4	24.9	26.1	27.0	degrees
z_1	230	240	235	250	m
z_2	875	900	880	900	m
T_0	4.23	4.86	5.31	5.77	dynes/cm²
$\mathring{v}$	15.77	16.66	17.51	17.95	m/sec
$\{T_y/v_y' - T_x/v_x'\}_m$	—18	—10	1	7	g/cm sec

In order to decide which α_0 is correct, a trial and error method was applied by computing the austausch-coefficient A with the aid of eq. (11 a) and eq. (11 b) from $A = T_x/v_x'$ and $A = T_y/v_y'$. Since A was defined as a scalar height function, both of these equations should result in the same A-values at any level.

The lowest line of Table 2 shows height-averages between 50 and 400 m of $T_y/v_y' - T_x/v_x'$, which would correspond to the difference of fictitious tensor components when $A_{xx} = T_x/v_x'$ and $A_{yy} = T_y/v_y'$. When A is a scalar, we should expect these differences to equal zero. It follows from Table 2 that the computation based on $\alpha_0 = 26.1^\circ$ yields a satisfactory fit. It is rather obvious that abandoning the condition of a scalar austausch must result in arbitrary families of (A_{xx}, A_{yy})-distributions unless T_0 and $\mathring{v}$ are fixed accurately by independent observations. For unique results, it is necessary to assume $A_{xx} = A_{yy}$, i.e. a scalar A. Consequently, the outer conditions of the "Leipzig Wind Profile" can be listed as follows:

$\mathring{v} = 17.51$ m/sec
$T_0 = 5.31$ dynes/cm²
$\alpha_0 = 26.1^\circ$
$$\left.\begin{aligned} \varrho &= 0.00125 \text{ g/cm}^3 \\ f &= 0.000140 \text{ sec}^{-1} \end{aligned}\right\} \varrho f = 1.425 \; 10^{-7} \text{ g/cm}^3\text{sec}$$

The formulation of the geophysical conditions of wind profile observations is incomplete when data about the hydrostatic stratification of the air and the characteristics of the roughness of the earth's surface is missing. MILDNER (1932) stated that the observations were taken in a uniform "warm" air mass and that no indication of convectional processes occurred from 0915—1615 hours during which period the 28 pibal ascents were carried out.

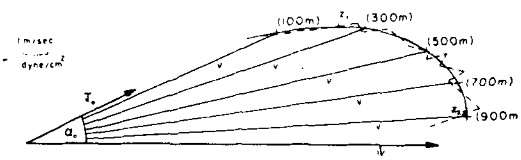

Fig. 2. The representative hodograph of the "Leipzig Wind Profile" from 50—900 meters. The broken line indicates MILDNER'S averaged observations.

In the same air mass, the 1400 hour sounding at the aerological station Lindenberg (less than 100 miles from Leipzig) showed a rather uniform lapse rate of $-0.65°$ C/100 m in the layer under consideration, which corresponds to an increase of potential temperature of $0.35°$ C/100 m.

The characteristics of the ground can be described by the fact that the ascents started over a grass-covered airfield while the surroundings were rather plain. MILDNER states that the wind passed over the city before the air arrived at the station which may have influenced the turbulence to an unknown degree.

MILDNER's examination of the weather maps resulted in a 2.5 mb/100 km horizontal pressure gradient which was independent of height in the lower 1 km layer; thus, $\mathring{v} = |\nabla p| / \varrho f = 17.5$ m/sec which equals our value in the above list.

The "Leipzig Wind Profile" is a rather unique and—at the present time—unparalleled example of a representative wind distribution in the frictional layer of the atmosphere. Therefore, and in order to stimulate further research, it appears justifiable to publish in Table 3 the representative wind data as elaborated from Mildner's graph. In the process of adjustment v_x, v_y, v_x' and v_y' were considered to be smooth height functions. The differences between original and adjusted or, representative wind components as illustrated by the hodograph (fig. 2) correspond to a standard deviation of approximately $\pm$ 0.05 m/sec; the small magnitude appears to be due to both the stable weather situation and the large number of observations.

From eqs. (11 a, b) it follows that in a scalar austausch theory

$$A = \sqrt{T_x^2 + T_y^2} / \sqrt{v_x'^2 + v_y'^2} = T/v'. \quad (15)$$

Another and previously unknown formula for the austausch-coefficient can be derived as follows. Regarding $\mathring{\mathbf{v}}' = 0$, eq. (2) can be written

$$\mathbf{T} = A\ (\mathbf{v} - \mathring{\mathbf{v}})' \quad (16)$$

Upon multiplying eqs. (1) and (16) by $i\ (\mathbf{v} - \mathring{\mathbf{v}})$ and $i\ (\mathbf{v} - \mathring{\mathbf{v}})'$, respectively, one obtains, when eq. (5 a) is considered,

$$i\,(\mathbf{v} - \mathring{\mathbf{v}})' \cdot \mathbf{T} = (i(\mathbf{v} - \mathring{\mathbf{v}}) \cdot \mathbf{T})' = -\varrho f (\mathbf{v} - \mathring{\mathbf{v}})^2 \quad (17)$$

Let s be the absolute value of $(\mathbf{v} - \mathring{\mathbf{v}})$ when β is the angle between $(\mathbf{v} - \mathring{\mathbf{v}})$ and $-\mathring{\mathbf{v}}$; then, $\tan \beta = -v_y/(v_x - \mathring{v})$ and

$$\mathbf{v} - \mathring{\mathbf{v}} = -\mathbf{i}\, s\, e^{-i\beta};$$

$$(\mathbf{v} - \mathring{\mathbf{v}})' = (\mathbf{v} - \mathring{\mathbf{v}}) \left(\frac{s'}{s} - i\,\beta'\right) \quad (18)$$

With the aid of eqs. (17), (18) and the boundary conditions (9) considering that $(i\,(\mathbf{v} - \mathring{\mathbf{v}}) \cdot \mathbf{T})_0 = -i\,\mathring{\mathbf{v}} \cdot \mathbf{T}_0 = -\mathring{v}\, T_0 \sin \alpha_0$, we obtain finally

$$A = \left(T_0\, \mathring{v} \sin \alpha_0 - \varrho f \int_0^z s^2\, dz\right) \Big/ s^2 \beta' \quad (19)$$

The three different expressions for the austausch-coefficient, eqs. (10), (15) and (19), result in A-values which are listed in Table 3. In view of certain numerical integration difficulties which arise from insufficient knowledge of wind distribution in the lowest 50 m-layer, the differences of the three A-series appear tolerable. The arithmetic average of the three A-distributions, therefore, may be called the "representative average" of the austausch-distribution from the "Leipzig Wind Profile". Table 3 shows also the standard deviation of the average. With reference to Table 1, MILDNER's A-values are of the true order of magnitude; however, they are too large below 300 m and too small above owing to incorrect assumptions with regard to the azimuth of the geostrophic wind.

It appears significant that the vertical distributions of A and v_y are rather similar to each other (see Table 3). This may indicate that the energy for maintaining austausch or turbulence in a steady frictional wind profile is taken from the potential energy of the horizontal pressure distribution by means of v_y, the steady flow from higher to lower pressure which, naturally, must reduce, and finally, eliminate the horizontal pressure differences or, the geostrophic motions, when these are not renewed by other atmospheric processes.

Fig. 2 shows the "Leipzig Wind Profile" as an illustration of an observed frictional wind spiral. Fig. 3 shows the corresponding stress spiral.

Finally, it may be noted that the direct integration of eq. (1) with regard to the boundary conditions at $z = \infty$: $T_\infty = 0$, $\mathbf{v}_\infty = \mathring{\mathbf{v}}$, yields

Table 3. The "Leipzig Wind Profile" and the subsequent austausch-distribution. z = height (m); v_x = wind component parallel to the geostrophic wind (m/sec); v_y = wind component perpendicular to the geostrophic wind (m/sec); A = austausch-coefficients (g/cm sec).

z	v_x	v_y	A eq. (15)	A eq. (10)	A eq. (19)	A representative average	A standard deviation
0	0.00	0.00	—	—	—	—	—
50	9.15	4.35	152	116	127	132	± 8
100	10.45	4.64	173	138	177	163	12
150	11.58	4.80	179	158	182	173	8
200	12.60	4.95	183	170	184	179	5
250	13.48	4.96	184	174	183	180	3
300	14.30	4.90	181	176	181	179	1
350	14.97	4.78	176	173	175	175	1
400	15.62	4.60	167	166	164	166	1
450	16.28	4.29	159	160	160	160	0
500	16.83	4.00	149	151	151	150	1
550	17.30	3.71	141	145	141	142	1
600	17.70	3.37	136	140	135	137	1
650	17.99	3.07	129	135	120	128	4
700	18.23	2.73	119	128	107	118	6
750	18.42	2.43	113	119	92	108	8
800	18.60	2.06	100	105	80	95	8
850	18.66	1.70	83	89	70	81	6
900	18.68	1.31	73	76	60	70	5
950	18.62	0.91	57	67	50	58	5

$$T_{0x} = \int_0^\infty \varrho f v_y \, dz \qquad (20\,a)$$

$$T_{0y} = -\int_0^\infty \varrho f (v_x - \mathring{v}) \, dz \qquad (20\,b)$$

These relations were used previously (ROSSBY and MONTGOMERY, 1935). However, they appear less practicable than eqs. (12 a) and (12 b) since the latter can be applied for a final and well-defined height interval in which both assumptions $\mathring{\mathbf{v}}' = 0$ and $\varrho' = 0$ appear more tolerable than for layers of undetermined height.

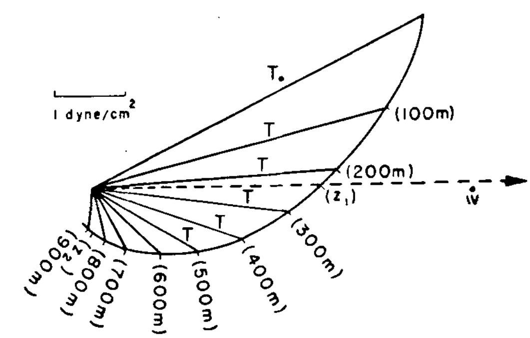

Fig. 3. The representative stress spiral of the "Leipzig Wind Profile". The significant levels z_1 and z_2 where the stress vector is parallel and perpendicular to the geostrophic wind vector are indicated in Figures 2 and 3.

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