The Vertical Distribution of Wind and Turbulent Exchange in a Neutral Atmosphere[1]

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Abstract. The problem of the approach to the geostrophic wind is solved assuming that the exchange coefficient is proportional to the one-third power of the rate of dissipation of turbulent energy. Reasonable agreement between the predicted and observed values of the surface stress, surface wind direction, and height of the layer of frictional influence is obtained when the size of the eddies is assumed to become independent of height at a relatively low level. The predicted vertical distribution of exchange coefficient is qualitatively in good agreement with that observed at Leipzig.

It is the general purpose of this paper to present a theory of the distribution of wind and turbulent exchange processes throughout the entire layer of frictional influence in the atmosphere and to compare some of the predictions of the theory with observations.

In approaching the problem it is expedient to simplify the atmospheric boundary layer by assuming that the mean motion is in a steady state and is entirely horizontal and uniform at every level, and that within this layer the density may be considered to be independent of height. The equations of horizontal motion then take the form

$$\begin{aligned} f(v - v_g) + \frac{d}{dz}\left[K_m \frac{d}{dz}(u - u_g)\right] &= 0 \\ -f(u - u_g) + \frac{d}{dz}\left[K_m \frac{d}{dz}(v - v_g)\right] &= 0 \end{aligned} \tag{1}$$

in which f is the Coriolis parameter, u and v are the horizontal components of the wind, u_g and v_g are the corresponding components of the geostrophic wind, and K_m is the kinematic exchange coefficient for momentum.

A number of theoretical wind distributions have been derived by specifying in various ways the vertical distribution of K_m. *Ekman's* [1905] classical solution, which was independently reached by *Taylor* [1915], applies when K_m is independent of height; it yields an equiangular spiral approach to the geostrophic wind with increasing height, but it is unsatisfactory near the ground. *Köhler* [1933], assuming K_m to increase as some power of the height, found solutions in the form of Bessel functions. *Ellison* [1956] and Kibel (see *Yudin and Shvetz* [1940]) have solved these equations using the functional form for K_m

$$K_m = ku^*z \tag{2}$$

in which k is von Kármán's constant and u^* is the friction velocity at the surface. This solution is in the form of a Hankel function and approaches a logarithmic form close to the surface, in agreement with the observed wind profiles near the surface during adiabatic conditions. However, the vertical distribution of the exchange coefficient is unrealistic, and the predicted relations between the surface wind speeds and directions and the geostrophic wind at different conditions of surface roughness do not agree with the observed values.

A rather complete theory of the wind distribution has been offered by *Rossby and Montgomery* [1935]. By combining a Prandtl-type surface layer with a logarithmic spiral solution reached earlier by *Rossby* [1932], the authors were reasonably successful in predicting the speed and direction of the surface wind from a knowledge of the roughness and pressure gradient at the surface. It appears that the pre-

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dictions of the angle between the surface wind and geostrophic wind are generally too small, but the proper interpretation of wind data is difficult. A similar type of patching together of a surface layer in which Kibel's solution is characteristic with an upper layer in which an Ekman solution prevails has been carried out by *Yudin and Shvetz* [1940].

The turbulent exchange coefficient K_m is actually not explicitly determined by external factors, but is implicitly determined by the wind distribution itself. Following a suggestion made by *Heisenberg* [1948], we suppose that K_m is related to the wind shear through the expression

$$K_m = \epsilon^{1/3} l^{4/3} \tag{3}$$

in which ϵ is the rate of dissipation of turbulent energy per unit mass and l is a length that describes the significant eddy size. The distribution of l in the atmosphere will be discussed later.

In the present circumstances the rate of dissipation of turbulent energy is given by

$$\epsilon = K_m\left[\left(\frac{du}{dz}\right)^2 + \left(\frac{dv}{dz}\right)^2\right] = K_m s^2 \tag{4}$$

where s is the magnitude of the wind shear. Upon substituting for ϵ in (3) we obtain

$$K_m = l^2 s \tag{5}$$

This is the expression used by *Prandtl* [1932] to explain the wind distribution close to the ground.

It will facilitate subsequent treatment of the equations of motion to express them in terms of nondimensional parameters. For this purpose, let the following quantities be defined:

$$\xi = \ln z/z_0 \tag{6}$$

$$G = (u_g^2 + v_g^2)^{1/2} \tag{7}$$

$$q = (u - u_g)/G \qquad r = (v - v_g)/G \tag{8}$$

$$S = \frac{kzs}{u^*} = \frac{kG}{u^*}\left[\left(\frac{dq}{d\xi}\right)^2 + \left(\frac{dr}{d\xi}\right)^2\right]^{1/2} \tag{9}$$

in which z_0 is the roughness parameter as used by Prandtl. It should be noted that u^* is defined in terms of the surface stress τ_0 by

$$u^* = (\tau_0/\rho)^{1/2} \tag{10}$$

and therefore, as used herein, is independent of height. We shall also make use of the dimensionless parameters

$$P = ku^*/fz_0 \tag{11}$$

$$b = u^*/kG \tag{12}$$

Equations 1 can then be transformed into the following set of equations by multiplying through by z/fz_0G and substituting for the dimensionless parameters defined above.

$$\begin{aligned} re^\xi + P\frac{d}{d\xi}\left(\frac{l^2 S}{k^2 z^2}\frac{dq}{d\xi}\right) &= 0 \\ -qe^\xi + P\frac{d}{d\xi}\left(\frac{l^2 S}{k^2 z^2}\frac{dr}{d\xi}\right) &= 0 \end{aligned} \tag{13}$$

At indefinitely great heights the momentum flux and its derivative approach zero. It follows that qz/z_0 and rz/z_0 approach zero at infinity. To specify the boundary condition at the ground simply, it is convenient to choose the coordinate axes in such a manner that $dr/d\xi$ vanishes at the surface. The x direction then coincides with the direction of the wind immediately above the surface. Then, if we specify that at $\xi = 0$ (i.e., at $z = z_0$) the wind speed is zero, we obtain one of the surface boundary conditions, namely:

$$\begin{aligned} q(0) &= -\cos\psi_0 \\ r(0) &= +\sin\psi_0 \end{aligned} \tag{14}$$

where ψ_0 is the (positive) angle between the wind just above the surface and the geostrophic wind. The other boundary condition follows from the fact that the downward momentum flux at the surface is ρu^{*2}, and therefore we have in the vicinity of the surface

$$u^* = ls \tag{15}$$

Since it is known that l equals kz close to the surface, we can express this condition as

$$S(0) = 1 \tag{16}$$

or

$$(dq/d\xi)_0 = b \qquad dr/d\xi = 0 \tag{17}$$

Because of the implicit variable S in equations 13, these equations are nonlinear and must be solved by numerical means. The numerical solution is facilitated by replacing the two second-order equations by a set of four equivalent equations:

$$dq/d\xi = k^2 z^2 R/Pl^2 S \tag{18}$$

$$dr/d\xi = k^2 z^2 Q/Pl^2 S \tag{19}$$

$$Q = \int_0^\xi qe^\xi \, d\xi \tag{20}$$

$$R_A = Pb - \int_0^\xi re^\xi \, d\xi \tag{21}$$

together with an equation which follows by applying the definition of S to the first two equations of this set:

$$S^2 = (k^2 z^2/l^2 Pb)(R^2 + Q^2)^{1/2} \tag{22}$$

As soon as the form of l is specified, the integration can be carried upward from the level $\xi = 0$ in small increments $\Delta\xi$, using the method of finite differences. The complete solution is determined, in principle, as soon as a value has been chosen for P, but the mechanics of finding the distribution of wind and exchange coefficient are hampered by the fact that appropriate values of the parameters b and ψ_0 are not initially known. Instead they are implicity determined by the boundary conditions at infinity, which may now be expressed by the requirement that R and Q must approach zero. The correct solution can be found by selecting trial values of b and ψ_0 and refining them by repeating the calculation until the upper boundary condition is considered to be sufficiently well satisfied.

The central problem lies in the specification of the distribution of the mixing length l. It is well known from Prandtl's surface-layer theory and observations that l is given by kz close to the surface. From a study of observations by *Mildner* [1932], Rossby and Montgomery concluded that l cannot continue to be proportional to distance from the ground at greater elevations. According to Rossby and Montgomery's theory l increases linearly up to a height of 12 per cent of the gradient level height, then decreases linearly to zero again at the gradient level. They found, however, that better agreement with observed winds can be obtained in periods of light wind if a 'residual austausch' of 50 g cm^{-1} sec^{-1} is added to that determined from their theory. This presence of a residual austausch suggests that l may maintain a significant magnitude at upper levels instead of returning to zero.

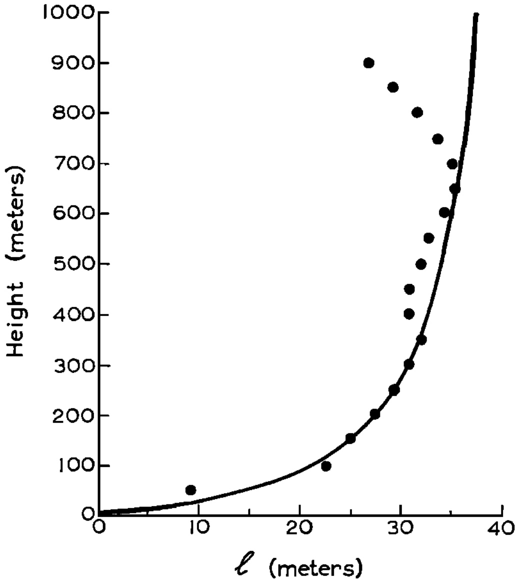

Fig. 2. Vertical distribution of the length l for *Lettau*'s [1950] analysis of the Leipzig wind profile. Calculated values (dots) are compared with the curve representing equations 24 and 25.

Before considering further the question of the vertical distribution of mixing length it is interesting to take up a suggestion by *von Kármán* [1930] that the length may be determined by the nature of the wind profile. The simplest way of doing this is by the expression

$$l = \frac{-ks}{ds/dz} \tag{23}$$

and it may be easily verified that in the surface layer the wind distribution is such that this specification is equivalent to that of Prandtl, namely, $l = kz$. At higher levels this equivalence no longer applies. Figure 1 shows the wind hodograph that results by integrating this case numerically by the procedure outlined above for a choice of P equal to 1612, corresponding to a geostrophic wind about 10 m sec^{-1} at Brook-

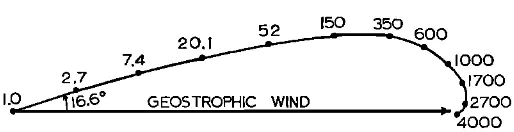

Fig. 1. Wind distribution for an extended von Kármán boundary layer model, corresponding to z_0 of 1 meter and G/f of 10^7 cm. Heights are indicated as the ratio z/z_0.

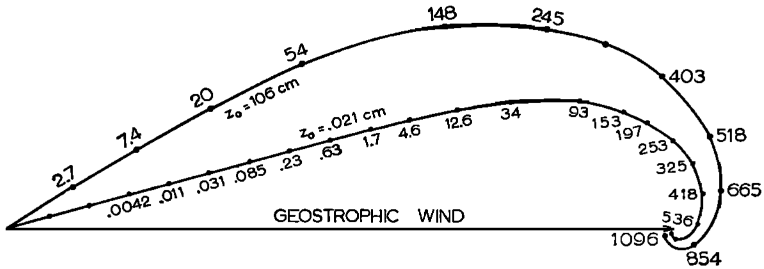

Fig. 3. Wind hodographs for rough and smooth surfaces corresponding to G/f equal to 10^7 cm. Heights are in meters.

haven, N. Y., where z_0 is about 1 meter. The indicated heights in units of z/z_0 may thus be interpreted as actual heights in meters. As is to be expected, the wind at first increases logarithmically with height and then approaches the geostrophic wind more slowly at great heights. The surface wind deviation angle ψ_0 is only half the value of 32° observed by *Bernstein* [1959] at Brookhaven under neutral barotropic conditions, and the predicted surface stress is too large in relation to the geostrophic wind. Also, the height of the gradient wind level appears to be about 4 times higher than is customarily observed in similar locations. Von Kármán's equation for l is not valid throughout most of the atmospheric boundary layer.

We have seen that it is necessary to limit the size of l in the free atmosphere. It is almost impossible to determine this distribution by direct observation, because the distribution is critically dependent on an exact knowledge of the geostrophic wind at all heights. The latter cannot be determined directly with sufficient accuracy, so recourse must be made to the wind profile itself. *Lettau* [1950] has described one way of doing this. Attempts by the author to use this and another method on observations from several different sources have yielded a bewildering variety of vertical distributions of l. The most common type of distribution is an increase up to 200 or 300 meters followed by a more or less constant value, usually of less than 100 meters, at higher levels. This kind of distribution results from Lettau's analysis of the Leipzig wind profile (Fig. 2). In agreement with this conclusion is the observation made by *Panofsky and*

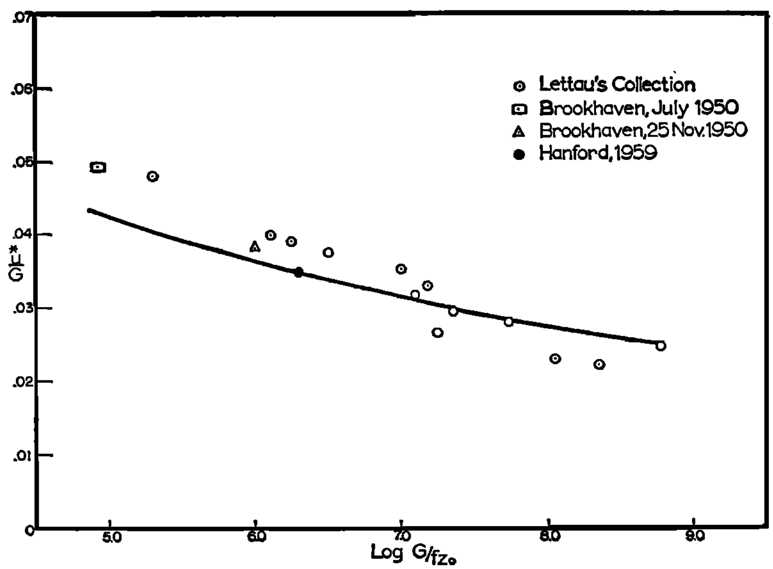

Fig. 4. Predicted and observed relationship between u^*/G and surface Rossby number G/fz_0.

McCormick [1960] from spectrum studies that the eddies tend to reach a limiting size at a height between 150 and 250 meters above the surface. Accordingly, experimental wind computations have been made assuming l to have the form

$$l = kz/(1 + kz/\lambda) \qquad (24)$$

so that l increases as kz close to the ground and approches a fixed value at greater heights.

The new parameter λ represents the value reached by l in the free atmsophere. In a completely neutral atmosphere its value must be determined by some combination of the fundamental parameters f, z_0, and G (or u^*) having the dimensions of a length. Since z_0 is intuitively ruled out as a factor affecting characteristics of the free atmosphere, a reasonable choice for length parameter is the ratio G/f. The proportionality constant has been determined in such a way as to yield a surface wind deflection angle of 32° at Brookhaven under neutral barotropic conditions with roughness parameter 1 meter and geostrophic wind of 10 m sec^{-1}, in accordance with the findings of *Bernstein* [1959]. The resulting relation

$$\lambda = 0.00027G/f \qquad (25)$$

has been employed for calculating the results described below. It fits the values computed from the Leipzig profile reasonably well, especially at lower levels where its effect on the wind profile is most critical. An alternative hypothesis is that λ is proportional to u^*/f; it is not equivalent, but seems a priori just as acceptable.

With l thus specified, (18) to (22) were transformed to central difference equations and programmed for solution on an IBM 650 digital computer. The calculations were done at intervals of 0.25 in ξ; thus each successive level rep-

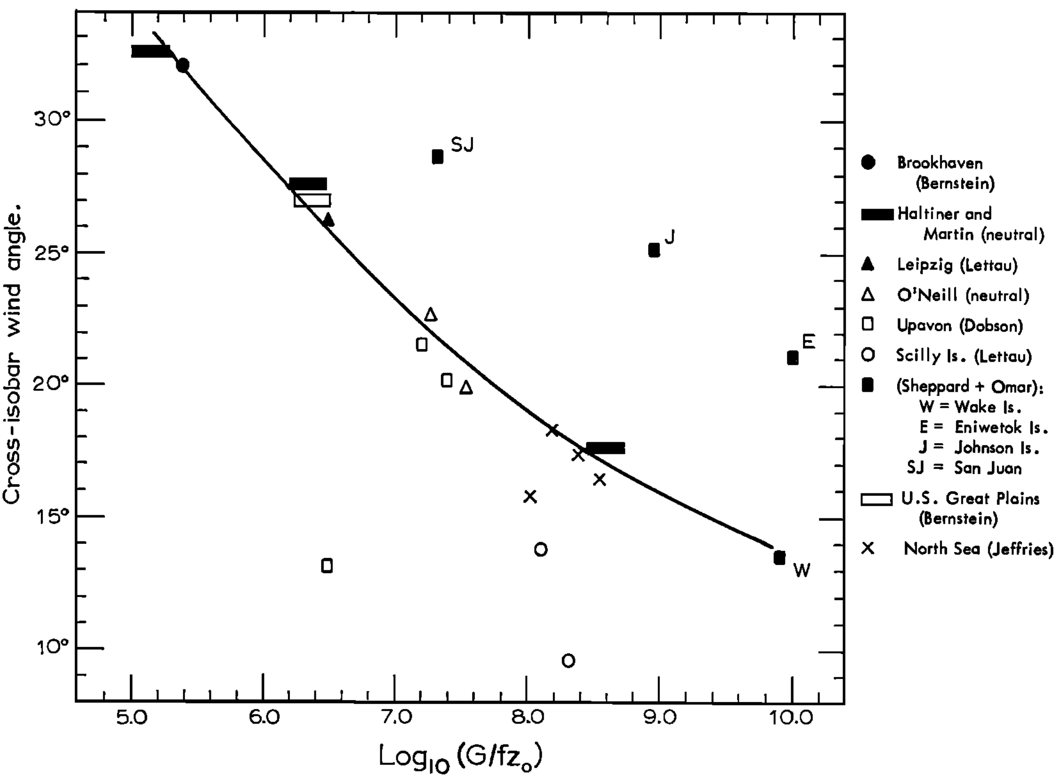

Fig. 5. Predicted and observed angle between the surface wind direction and surface isobars. The data have been obtained from *Bernstein* [1959], *Dobson* [1914], *Jeffries* [1920], *Lettau* [1950, 1957], *Sheppard and Omar* [1952], and the author.

resents an increment of z/z_0 in the ratio of 1.284. Each trial solution was carried up to the level where S began to increase, and, by means of repeated trials, values of b and ψ_0 were refined until they were accurate to 0.0001 and 0.1 degree respectively. The number of levels required for each solution varied from 32 in the roughest situations to 66 in the smoothest ones. In every case the number of levels was sufficient to reach or exceed the gradient wind level where the wind direction becomes parallel to the geostrophic wind.

The wind hodographs for two contrasting sets of conditions are illustrated in Figure 3. In both the wind is, for all practical purposes, constant in direction up to 20 meters and is indistinguishable from the Prandtl logarithmic distribution, as is to be expected from the boundary conditions that were imposed. The surface wind direction is 32° to the left of the isobars in the rough situation and is reduced to 14° in the smooth one. At higher levels the wind vector spirals in toward the geostrophic wind in a way qualitatively similar to the Ekman spiral. However, in comparison with the latter, the rate of approach is more rapid at the highest levels.

The surface wind stress in relation to the environmental parameters is shown in Figure 4. Since b is a function of P it is also a function of the ratio P/b. It is therefore possible to predict the ratio u^*/G as function of the nondimensional number G/fz_0, which was first suggested by *Rossby and Montgomery* [1935]. The small systematic discrepancies may be due in part to the presence of horizontal temperature gradients in the natural atmosphere. The observations for small G/fz_0 were obtained in temperate latitudes with relatively small geostrophic winds. In these situations the increase of pressure gradient with height tends to increase the wind shear and promotes a larger surface wind stress than would be expected in a barotropic atmosphere. The observations for the extremely large Rossby numbers came from the trade-wind region where the pressure gradient decreases with elevation.

In Figure 5 the predicted relationship between ψ_0 and the surface Rossby number is compared with data collected from several sources. The large scatter in the observational data is partly a reflection of the sensitivity of the surface wind direction to horizontal temperature gradients in the manner shown by *Bernstein* [1959]. When the pressure gradient increases with height the wind angle decreases, and this effect is most noticeable when the pressure gradient at the surface is weak. Thus we may explain why the angles for the feeblest pressure gradients in both Dobson's and Jeffries' data are systematically too small. The data by Sheppard and Omar from the trade-wind stations are probably affected in the opposite manner. Bernstein's data for Brookhaven and the Great Plains are selected for neutral barotropic conditions,

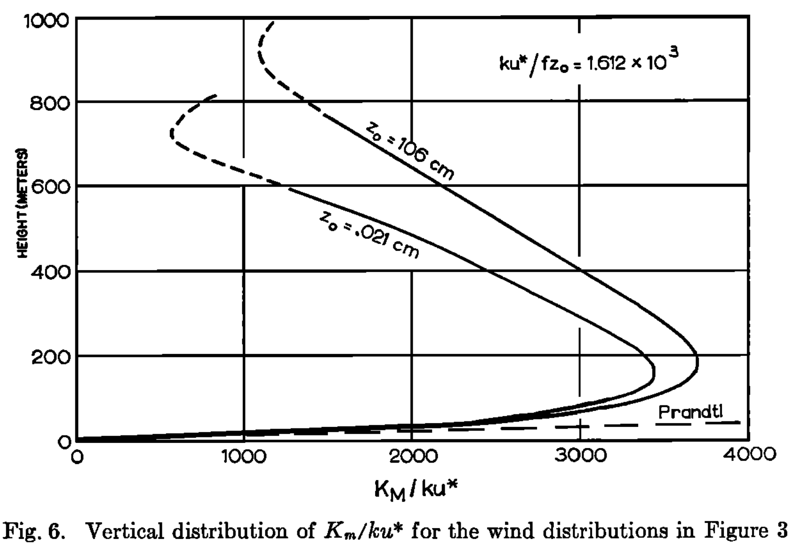

Fig. 6. Vertical distribution of K_m/ku^* for the wind distributions in Figure 3.

and the Leipzig case is also barotropic. The O'Neill data are neutral up to at least 1000 meters. Unfortunately, it is not possible to ascertain the degree by which the remaining observed angles are systematically affected by baroclinicity and stability. However, they represent moderate or large pressure gradients, and so are not likely to be very seriously affected.

The vertical distributions of exchange coefficients that accompany the wind solutions have been computed by applying a modified form of equation 5:

$$K_m/ku^* = Sz/(1 + kz/\lambda)^2 \qquad (26)$$

Those applying to the two wind distributions in Figure 3 are illustrated in Figure 6. These curves confirm the distributions found at Leipzig and elsewhere in that they increase to a maximum at a height of a few hundred meters and then decrease to small values at higher levels. The increase in computed values above 600 meters is due to failure to completely satisfy the boundary condition at infinity; in reality the exchange coefficient, like S, must return to zero at infinity. It is interesting that the height of the maximum exchange coefficient shows almost no variation over the entire range of surface roughness normally encountered. According to the hypothesis stated in equation 25, however, this height appears to be at least roughly proportional to the geostrophic wind speed. Thus the height found for the maximum exchange at Leipzig by Lettau, 250 meters, agrees quite well with the expectation of the theory for the geostrophic wind speed of 17 m sec^{-1} estimated by *Lettau* [1950].

Finally we may consider the height of the gradient wind level, which we define as the level where the direction of the wind first becomes parallel to the direction of the geostrophic wind. The value of the ratio Zg/z_0 at which this condition occurs is a function only of G/fz_0, and this function has been determined from the numerical solutions. The gradient wind level is well represented by the equation

$$\log_{10} z_g = 0.008 \log_{10} z_0 + 0.992 \log_{10} (G/f) - 2.186 \qquad (27)$$

The height of the gradient level increases very slowly with increasing surface roughness, and is very nearly proportional to the ratio G/f. This behavior is qualitatively similar to that predicted by *Rossby and Montgomery* [1935] and appears to be in agreement with experience, but there are no observations sufficiently accurate to provide a critical test of these conclusions.

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