Consult your midterm review sheets for the first two-thirds of the class.

**Momentum Equations**

\[
\frac{du}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u + 2\Omega v \sin \phi - 2\Omega w \cos \phi + \frac{uw \tan \phi}{a_E} - \frac{uw}{a_E}
\]

\[
\frac{dv}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v - 2\Omega u \sin \phi - \frac{u^2 \tan \phi}{a_E} - \frac{vw}{a_E}
\]

\[
\frac{dw}{dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - g + \nu \nabla^2 w + 2\Omega u \cos \phi + \frac{u^2 + v^2}{a_E}
\]

**Scale Analysis**

Simplify equations based on synoptic scales of motion. Letting \( f = 2\Omega \sin \phi \),

\[
\frac{du}{dt} = fv - \frac{1}{\rho} \frac{\partial p}{\partial x}
\]

\[
\frac{dv}{dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]

\[
\frac{1}{\rho} \frac{\partial p}{\partial z} = -g
\]

\[\nabla \cdot (\bar{\rho} \bar{u}) = 0\]

**Synoptic Scale Equations in Pressure Coordinates**

\[
\frac{du}{dt} = fv - \frac{\partial \Phi}{\partial x}\bigg|_p
\]

\[
\frac{dv}{dt} = -fu - \frac{\partial \Phi}{\partial y}\bigg|_p
\]

\[
\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho}
\]

\[\nabla \cdot \bar{u} = \frac{\partial u}{\partial x}\bigg|_p + \frac{\partial v}{\partial y}\bigg|_p + \frac{\partial \omega}{\partial p} = 0\]

where the total derivative in pressure coordinates becomes \( d/dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + \omega\partial/\partial p \). Partial derivatives with respect to \( x, y, t \) are taken at constant pressure.
Natural Coordinates

Define unit vector $\hat{t}$ to be parallel to the horizontal velocity, and unit vector $\hat{n}$ to be normal and to the left of the horizontal velocity. Synoptic scale momentum equations become:

$$\frac{dV}{dt} = -\frac{\partial \Phi}{\partial s}$$

$$\frac{V^2}{R_{\text{curv}}} + fV = -\frac{\partial \Phi}{\partial n}$$

$R_{\text{curv}} > 0$ for counterclockwise flow, $< 0$ for clockwise flow.

Balanced Flows

Parcel does not change speed and flow parallel to height contours, so left with:

$$\frac{V^2}{R_{\text{curv}}} + fV = -\frac{\partial \Phi}{\partial n}$$

Rossby Number

Taking ratio of local centrifugal acceleration to Coriolis acceleration in natural coordinates yields the Rossby number:

$$Ro = \left| \frac{V}{fR_{\text{curv}}} \right|$$

Geostrophic Balance


$$V_{\text{geostrophic}} = -\frac{1}{f} \frac{\partial \Phi}{\partial n}$$

Cyclostrophic Balance


$$V_{\text{cyclostrophic}} = \sqrt{-R_{\text{curv}} \frac{\partial \Phi}{\partial n}}$$

Gradient Wind Balance

$Ro \approx 1$. Centrifugal force about the same size as the Coriolis force. Balance between centrifugal force, Coriolis force, and pressure gradient force.

$$V_{\text{gradient}} = -\frac{fR_{\text{curv}}}{2} + \sqrt{\frac{f^2 R_{\text{curv}}^2}{4} - R_{\text{curv}} \frac{\partial \Phi}{\partial n}}$$
Force balance diagrams for all three balanced flows for high/low pressure (geopotential height) centers and for clockwise/counterclockwise flow.
1a) Consider a parcel in a circular polar vortex centered at the North Pole. The parcel has a zonal velocity of 50 m s$^{-1}$ and is located at 60°N. Determine its Rossby number.

1b) Assume the flow is balanced. Given your answer in 1a), describe and sketch the horizontal balance of forces in the polar vortex at 60°N.
2a) An island in the northern hemisphere heats up during the day. Since the ocean around the island has a higher heat capacity, the column of air over the ocean is cooler than the column of air over the island. The schematic below is a zonal cross section ($x$-$z$) through the island, sketch a resonable pressure surface above $p_1$ and state your reasoning for why the pressure surface must look like this.
2b) Based on the pressure surface you drew, would there be high or low geopotential heights over the island compared with the ocean? A circulation sets up over the island that has a negative radius of curvature and is in gradient wind balance with the geopotential height distribution. Sketch and label the **horizontal** balance of forces of this circulation in the space below.
3a) Calculate the change in geopotential following a parcel if there is a westerly wind of 20 m s\(^{-1}\) and a southerly wind of 5 m s\(^{-1}\). The geopotential increases 10 m\(^2\) s\(^{-2}\) per 100 km to the east and decreases 40 m\(^2\) s\(^{-2}\) per 100 km to the north.

3b) Explain why the flow must be balanced, given your answer in 3a).