1) The Greenhouse Effect: Radiative equilibrium model for a single layer atmosphere

Simple models are often applied in climate sciences as a tool for illustrating fundamental physical processes. These models should not be considered as compact description of the complex real world. Rather they help us scientists to understand the complex world a little better. If we accept this concept then we must be willing to accept its limitation, too. In the following description it is the primary goal to provide the basic idea why the atmosphere increases surface temperatures on Earth. Let's start with the simplest model that we can use for this purpose. And let's practice to work with the equations to quantify the greenhouse effect in this simple global radiative energy model with a single atmospheric layer. This layer acts like an absorber and emitter of infrared radiation. But it's greenhouse effect is measurable only if we define a reference climate state. This reference system provides a surface temperature for comparison. Here we use as a reference climate system a planet without atmosphere.

A single parameter will describe the emissivity of the layer in the infrared wavelength range. Note that we use the physical concept that gray-body radiative emissivity is a fraction of the blackbody emissivity, and that the atmopshere absorbs in the same proportion the incoming radiation as it emits the radiation. The incoming sunlight and longwave (LW) radiative fluxes are further considered as the integral over the visible range (shortwave, SW) and a second integral over the LW spectrum in the infrared range. The atmosphere-free planet is given the same planetary albedo as the Earth: we assume 30% of the SW radiation is reflected back to space. The rest is absorbed at the surface and used to heat up the planet. In the equilibrium state incoming and outgoing radiation must cancel each other. Now this atmosphere-free system has only one freely adjustable climate variable, the global mean surface temperature T_s. This temperature we equate with the blackbody temperature at which the surface emits IR radiation back to space. So the problem is fairly simple. On the one hand we have the incoming radiation in the SW range $Q_0/4 = 1/4 * 1360\mathrm{Wm}^{-2} = 340\mathrm{Wm}^{-2}$, and with a fixed and known albedo $a = 0.3$ this gives a net incoming flux

$$F_{in} = \frac{Q_0}{4}(1-a) = 340\mathrm{Wm}^{-2} * 0.7 = 238\mathrm{Wm}^{-2} \tag{1}$$

We need an equation that relates now the surface equilibrium temperature with this net incoming radiative flux. Noting again that in equilibrium the outgoing LW radiative flux balances the incoming flux ($F_{in} = F_{out}$) and recalling that the Stefan-Boltzman law gives us the relationship for a blackbody radiation and the body's temperature our equation becomes

$$F_{in} = \sigma T_s^4 \tag{2}$$

This assumes that in the infrared the planet emits as a blackbody. The Stefan-Boltzmann constant $5.67 * 10^{-8}\mathrm{Wm}^{-2}\mathrm{K}^{-4}$ is given, and therefore we can solve for T_s:

$$T_s = \left[\frac{F_{in}}{\sigma}\right]^{1/4} = 255\mathrm{K} \tag{3}$$

The equilibrium surface temperature of the planet without atmosphere would be -19°C.

2) Adding the single atmospheric layer (window-gray approximation)

With a single layer atmosphere the planet is wrapped into an invisible blanket. Visible light is assumed in this simple case to pass through the atmosphere without any scattering or absorption. Planetary albedo does remain the same (0.3) and surface net shortwave flux is the same as at the top of the atmosphere. The atmosphere only affects the IR radiative flux. A part of it is absorbed and heats the atmosphere, another part is transmitted and escapes into space. But introducing the atmosphere means we have another parameter and one more climate variable: the emissivity ϵ and the atmosphere's temperature T_a, respectively. For the moment we leave the emissivity unspecified, but it is noted here that ϵ is a number between 0 and 1. For 1 the atmosphere becomes a blackbody and all IR that enters the atmosphere is absorbed. Hoewever, the atmosphere would also act as an emitter of radiation following the Stefan-Boltzmann law. For $\epsilon = 0$ the atmosphere would allow 100% of the IR coming from the surface to be transmitted and to escape to space. We assume ϵ is somewhere between these extremes.

In order to understand the greenhouse effect from this atmospheric layer, consider that the system had enough time to find its equilibrium. The most important idea in this conceptual model is that the atmospheric layer is thin but it has two emitting surfaces: one facing outer space and the other facing the surface. Both surfaces emit at the same temperature T_a with the same emissivity ϵ. One side sends radiation out to space the other back to the surface in equal amounts.

In this system there is a permanent exchange of radiative energy between surface and atmosphere (typical for systems that exchange fluxes), but all are in balance in an equilibrium state with constant temperatures. The radiative balance is given when the incoming radiative flux F_{in} is balanced by the LW radiation emitted to space (F_{out}). Compared to the first case (our reference climate system) there are two components in the balance equation contributing to F_{out}: A fraction of the surface radiative flux escapes through the atmosphere, and then the atmosphere's upward flux; both depend on ϵ. Likewise the Earth receives now some extra LW radiative flux from the atmosphere in addition to the SW flux.

Let's write this budget down for the top of atmosphere outgoing (LW) flux and for the surface (for the equilibrium state $F_{in} = F_{out}$; hence we use F_{in} in the equations below, only):

$$F_{in} = \epsilon \sigma T_a^4 + (1 - \epsilon)\sigma T_s^4 \tag{4}$$

$$F_{in} + \epsilon \sigma T_a^4 = \sigma T_s^4 \tag{5}$$

Two equations with two climate variables allows us to solve for the surface temperature and air temperature. Subtracting equation 4 from 5
gives us

$$\epsilon \sigma T_a^4 = \sigma T_s^4 - \epsilon \sigma T_a^4 - (1 - \epsilon)\sigma T_s^4 \tag{6}$$

Rearranging the terms that are left (note the canceling term σT_s^4 on the right hand side), leaves a simple equation relating surface and atmosphere's temperature.

$$2\epsilon \sigma T_a^4 = \epsilon \sigma T_s^4 \tag{7}$$

One thing we can read immediately from this equation: atmosphere is cooler than the surface temperature. That makes sense.

Now we want to know how much the greenhouse effect has warmed the surface compared with our first planet.

For this we can go back to Eq. 5 and substitute the atmospheric temperature term according to 7:

$$F_{in} = (1 - \frac{\epsilon}{2})\sigma T_s^4 \tag{8}$$

Resolving for T_s gives us:

$$T_s = \left[\frac{F_{in}}{(1-\frac{\epsilon}{2})\sigma}\right]^{1/4} \tag{9}$$

Now we can compare this solution with the previous atmosphere-free temperature solution from Eq. 3. First, let's check if the equation for the model with the window-gray atmosphere is consistent with Eq. 3: If the emissivity goes down to zero, the atmosphere becomes more and more transmittive in the IR range. Less and less radiative energy flux can heat the atmosphere and the emitted IR radiation going back to the surface goes down, too. For $\epsilon = 0$ the atmosphere would have no greenhouse effect and both equations are exact the same.

Now if we on the other hand assume that the IR emissivity is high (e.g. 0.78) then hardly any IR emitted from the surface escapes to space, most is aborbed in the atmosphere and helps to increase its temperature. This leads to emission of more IR to space and back to surface and helps to heat up the surface temperature. For that particular value of ϵ we would get a surface temperature of about 288K (15°C).

Note: This example was taken from Coakley and Yang "Atmospheric Radiation" Section 1.4. I emphasized here the intermediate steps.