The purpose of this assignment is to give you a hands-on appreciation for the concepts of stability analysis using the Lorenz 63 model. This model is given by the following equations:

\[
\begin{align*}
\frac{dx}{dt} &= -\sigma x + \sigma y \quad (1) \\
\frac{dy}{dt} &= -xz + rx - y \quad (2) \\
\frac{dz}{dt} &= xy - bz \quad (3)
\end{align*}
\]

where \( r, b \) and \( \sigma \) are all free parameters (typically set to 28.0, 8/3, and 10.0, respectively). Go to the course webpage and download the basic python code for this experiment. You are free to use another program to complete this assignment; however, you will be responsible for making sure it works correctly. For this analysis, concentrate on the x-z plane results. Your assignment should include a written answer for each question (1-2 paragraphs) and appropriate figures.

(1) Run the model in the default configuration we have used in class. We will call this the control trajectory of the model. For a variety of points along the control trajectory, compute the propagator matrix (M) for a 1.0 time unit forecast (hereafter denoted \( t_f \)) using the code provided. At each of these points, use the propagator matrix to predict the forecast error via the Euclidean norm for different sized isotropic initial condition errors (i.e., change the magnitude of the error in variable ic_error). Next, compute the forecast error with the non-linear model by adding the isotropic error to the control trajectory and integrating the nonlinear model forward. How does the linear prediction of the forecast error compare to the actual error? Are there locations within the state space volume where the linear model errors disagree with the nonlinear model? What does that say about the Lorenz 63 model at these points?

(2) Repeat the above calculations from above, but change the amount of time used to compute the propagator matrix (i.e., decrease and increase \( t_f \) with variable fcst_len). How does this change your results from above?

(3) Linear time-independent propagators can have a limited usefulness for a nonlinear model. For
In this step, compute the tangent linear model by changing the frequency over which the propagator matrix \( M \) is computed, which in turn will change the accuracy of the tangent linear approximation, by adjusting \( \text{tl}_\text{freq} \). Note that a value of 1 means computing the tangent linear model every timestep, a value of 10 means compute it every 10 timesteps. Repeat the process you used in (1), but where you hold the initial condition error at 0.1 units. Compare the difference between the linear and non-linear error when you compute the tangent linear model every 1, 2, 5, and 10 timesteps. How does your answer vary depending on location along the state space?

(4) We are now going to look at the growth of initial condition errors in this model using singular vectors. At several points along the model trajectory, compute the propagator matrix for a 1.0 time unit forecast, then compute the singular vectors, which are obtained by taking the eigenvalue decomposition of \( M^T M \). Plot the evolution of the first and second singular vector in the x-z space at various points along the trajectory (you can plot the evolution of singular vectors by setting \( \text{plot} \_\text{sv} = \text{True} \), and you can switch between the first and second singular vectors by changing \( \text{sing} \_\text{val} \) from 1 to 2). How does the orientation of the first and second singular vector change for different locations in the model trajectory? What is the orientation of the first and second singular vectors relative to each other for different points along the model trajectory? Explore the distribution of singular values along the state space trajectory. What locations are characterized by the largest values? What do these large values mean about the Lorenz 63 model?

(5) Compare the forecast error for a forecast that is initialized with an error consistent with the first singular vector with the isotropic error case? What does this result say about the initial-time singular vectors vs. an error that is uniform in all directions (i.e., isotropic)?

(6) Finally, one can use the adjoint of the tangent linear model to estimate the sensitivity of a forecast metric to the initial conditions, which is often referred to as adjoint sensitivity. Here, we are going to compute the sensitivity of the 1.0 time unit \( z \) coordinate forecast to the initial conditions of \( x, y, z \). Compute the sensitivity at various points along the control trajectory. At what point(s) is the forecast most sensitive to the initial conditions (i.e., the magnitude of the sensitivity is largest)? In general, the \( z \) forecast is most sensitive to which initial condition coordinate? How does the forecast error from a forecast initialized with an initial condition error consistent with the initial condition sensitivity compare to the forecast initialized with isotropic errors?