The QG Height Tendency Eqn

- can combine the QG vorticity and thermodynamic equations to get the QG Height Tendency Equation
  → an equation that predicts, based primarily on vorticity advection and differential temperature advection, whether geopotential heights will rise or fall at a given location

  → can use this eqn to predict 500 hPa height changes without a numerical model!

- in a previous lecture, derived the QG Vorticity Eqn:

$$\nabla_p^2\left[\frac{\partial \Phi}{\partial t}\right] = -f_0 \vec{V}_g \cdot \nabla\left[\frac{1}{f_0}\nabla_p^2 \Phi + f\right] + f_0^2 \frac{\partial \omega}{\partial p} + f_0 k \cdot \nabla \times \vec{F}$$

- from last lecture, derived the QG Thermodynamic Eqn

$$\frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) = -\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) + \omega\sigma + \frac{\alpha}{T}\frac{1}{c_p}\frac{dQ}{dt}$$

- need to put QG Thermo. eqn in different form so it can be subtracted from QG vorticity eqn:
  → multiply each term by $\frac{f_0^2}{\sigma}$
  → take $\frac{\partial}{\partial p}$

get

$$-\frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left(\frac{\partial \Phi}{\partial t}\right) = \frac{\partial}{\partial p}\frac{f_0^2}{\sigma}\left[-\vec{V_g}\cdot\nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \frac{\partial}{\partial p} f_0^2 \omega$$

$$+ \frac{\partial Q}{\partial p}$$

where Q is redefined for purpose of simplificatio

$$Q \equiv \frac{f_0^2 \gamma}{\sigma T c_p}\frac{dQ}{dt}$$

'note that the LHS of both equations has $\frac{\partial \Phi}{\partial t}$ term

$$\rightarrow \frac{\partial \Phi}{\partial t} \equiv \chi \equiv \text{height tendency}$$

'subbing in χ for $\frac{\partial \Phi}{\partial t}$, can rewrite both the QG thermodynamic and QG vorticity equations:

$$\nabla_p^2 \chi = -f_0 \vec{V_g}\cdot\nabla\left[\frac{1}{f_0}\nabla_p^2\Phi + f\right] + f_0^2\frac{\partial \omega}{\partial p} + f_0\hat{k}\cdot\nabla\times\vec{F} \quad ①$$

(QG vorticity Eqn)

$$-\frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\chi = \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right] + f_0^2\frac{\partial\omega}{\partial p} + \frac{\partial Q}{\partial p} \qquad (2)$$

(QG thermodynamic Eqn.)

now, subtract (2) from (1):
get

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi = \overbrace{-f_0\vec{V}_g\cdot\nabla\left[\frac{1}{f_0}\nabla_p^2\Phi + f\right]}^{A}\ \overbrace{-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\cdot\nabla\left(-\frac{\partial\Phi}{\partial p}\right)\right]}^{B}$$

$$\underbrace{-\frac{\partial Q}{\partial p}}_{C} + \underbrace{f_0\hat{k}\cdot\nabla\times\vec{F}}_{D}$$

→ QG Height Tendency Equation

Physical Interpretation of Terms

start with term on LHS:

$$\left(\underbrace{\nabla_p^2} + \frac{f_0^2}{\sigma}\underbrace{\frac{\partial^2}{\partial p^2}}\right)\chi \qquad \chi \equiv \frac{\partial\Phi}{\partial t}$$

involves 2nd derivatives of χ

- assume χ varies sinusoidally in x, y, and p:

$$\chi = \chi_0 \sin(kx)\cos(ly)\cos\left(\frac{\pi p}{p_0}\right)$$

$\chi_0 \rightarrow$ amplitude of χ

$\sin(kx) \rightarrow$ variation in x

$k = 2\pi / L_x \rightarrow$ zonal wavenumber

$L_x \rightarrow$ zonal wave length

$\cos(ly) \rightarrow$ variation in y

$l = 2\pi / L_y \rightarrow$ meridional wavenumber

$L_y \rightarrow$ meridional wave length

$\cos\left(\frac{\pi p}{p_0}\right) \rightarrow$ variation in p

- assume 180° phase shift from surface to tropopause

$p_0 = 1000$ hPa

- so can take $\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}$ of χ

$\rightarrow$ end up with

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi = -\left(k^2 + l^2 + \frac{f_0^2}{\sigma}\frac{\pi^2}{p_0^2}\right)\chi$$

$\rightarrow$ LHS is proportional to $-\chi$

- so if RHS (the <u>forcing</u> for height falls/rises) is positive,

  then $\chi \left(\equiv \frac{\partial \Phi}{\partial t}\right)$ is negative

  → geopotential height will decrease if forcing (can be from any or all terms on RHS) is positive!

- now, what about the individual forcing terms on RHS?

- term A:

$$\left(\nabla_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi \propto -f_o\vec{V_g}\cdot\nabla\Big[\overbrace{\frac{1}{f_o}\nabla_p^2\Phi}^{\zeta_g} + f\Big]$$

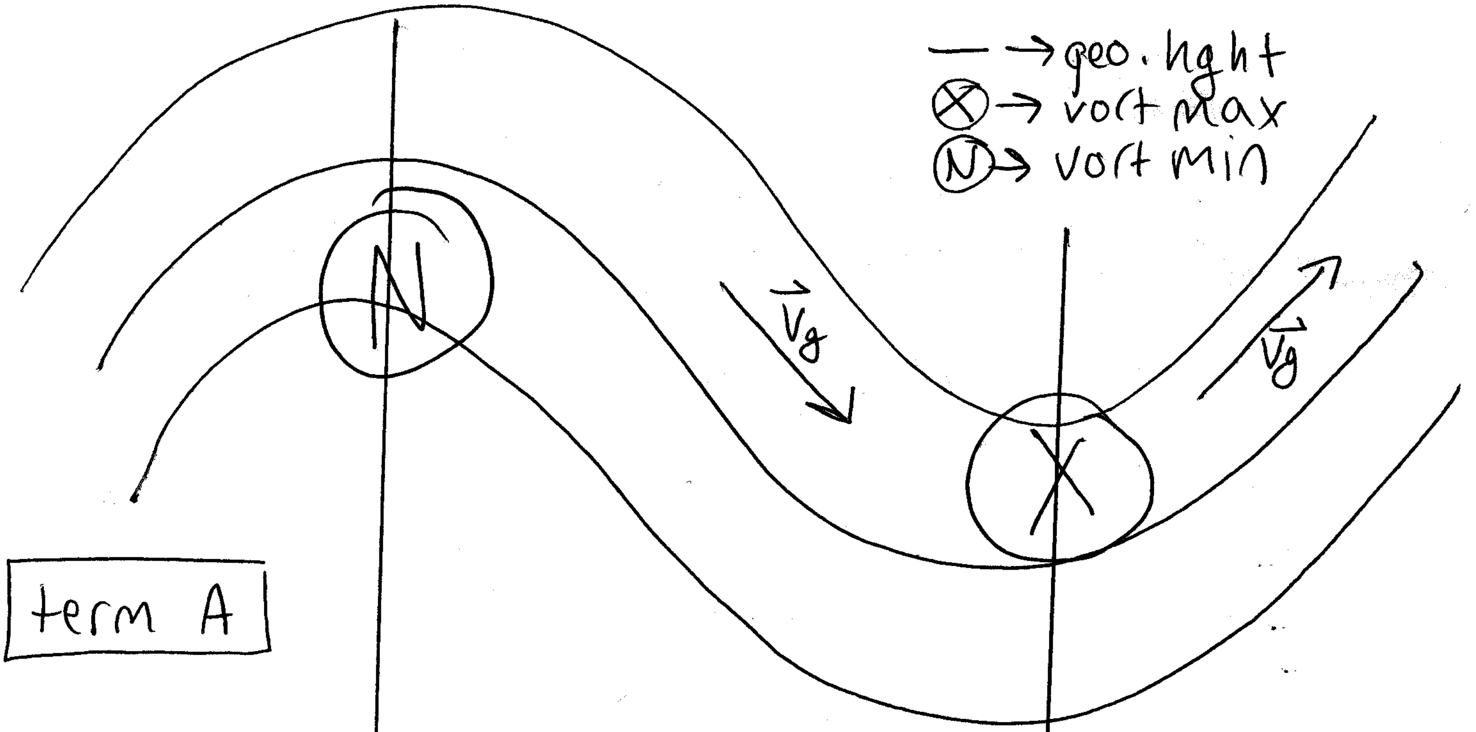
or

$$\propto -f_o\vec{V_g}\cdot\nabla\zeta_g - f_o\vec{V_g}\frac{\partial f}{\partial y}$$

- vorticity advection term (relative geostrophic vorticity and planetary vorticity)

  $\zeta_g + f \equiv$ absolute vorticity

- these terms tend to oppose each other in westerlies



$\boxed{\text{term A}}$

relative geostrophic vorticity advection contribution	$-f_0 \vec{V_g} \cdot \nabla \zeta_g < 0$ $\ominus \oplus \oplus \quad \oplus$ → $\chi > 0$ → height rises	$-f_0 \vec{V_g} \cdot \nabla \zeta_g > 0$ $\ominus \oplus \oplus \quad \ominus$ → $\chi < 0$ → height falls
planetary vorticity advection contribution	$-f_0 v \frac{\partial f}{\partial y} > 0$ $\ominus \oplus \ominus \quad \oplus$ → $\chi < 0$ → height falls	$-f_0 v \frac{\partial f}{\partial y} < 0$ $\ominus \oplus \oplus \quad \oplus$ → $\chi > 0$ → height rises

- relative geostrophic vorticity advection wants to move pattern <u>eastward</u>
- planetary vorticity advection wants to move pattern <u>westward</u>

· what is planetary, relative vorticity advection at base of trough and crest of ridge?

→ ZERO!

(v is zero, so no planetary vort. advection, and $\nabla \zeta_g$ is zero, so no relative vort. advection)

Implication: vorticity advection can only move troughs and ridges – it cannot strengthen or weaken them!

· term B:

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi \propto -\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]$$

thickness

→ differential thickness advection

· remember that $-V_g \cdot \nabla\left[-\frac{\partial \Phi}{\partial p}\right] > 0 \rightarrow$ WAA

$\sim T$

· if WAA decreases with height,

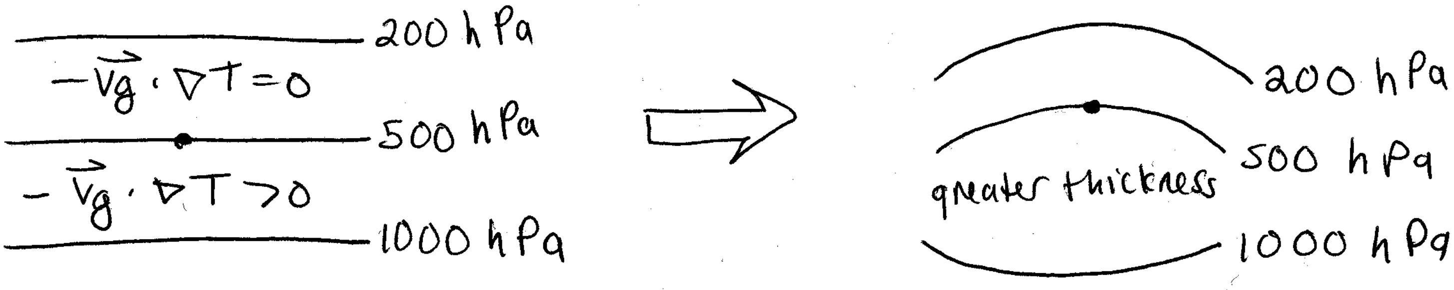
$$-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0$$

→ $\chi > 0$

→ geopotential heights increase at midlevels

<u>example</u>: low-level warm air advection

$t = 0$ $\qquad\qquad$ $t = \Delta t$



· what about for CAA decreasing with height?

$$-V_g \cdot \nabla\left[-\frac{\partial \Phi}{\partial p}\right] < 0 \rightarrow CAA$$

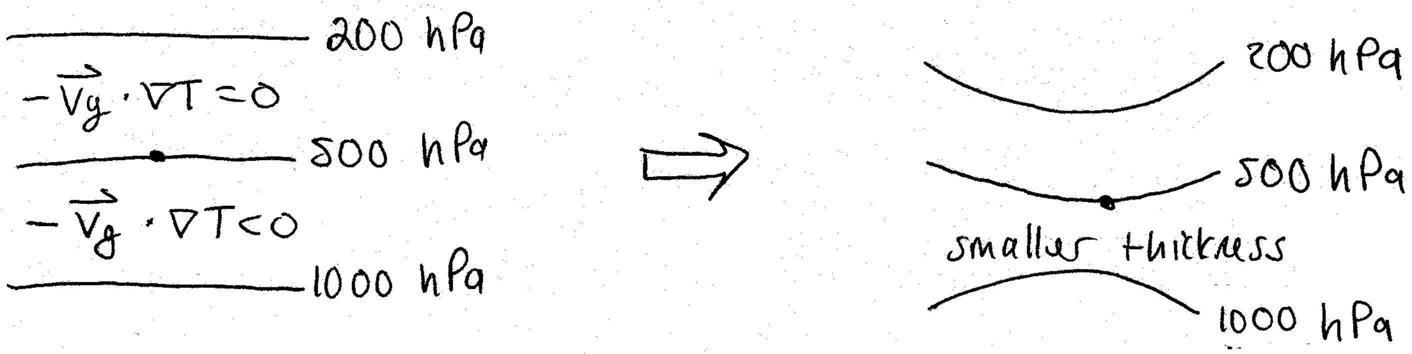
· if CAA decreases with height

$$-\frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V_g} \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0$$

→ $\chi < 0$

→ geopotential heights decrease at midlevels example: low-level cold air advection

$t=0$ $\qquad\qquad\qquad\qquad$ $t=\Delta t$

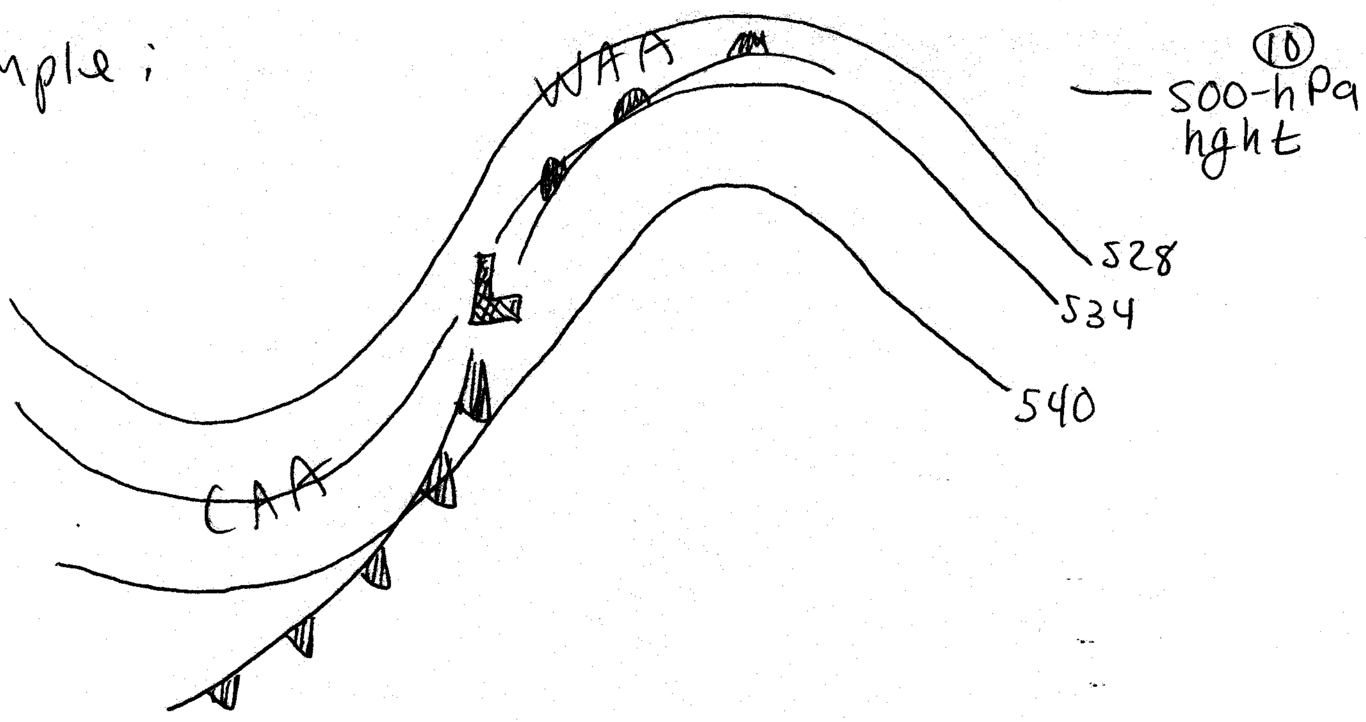


- if WAA occurs at upper levels instead of lower levels, heights fall at midlevels instead of rise!
- similarly, if CAA occurs at upper levels, heights rise at midlevels!

→ where the thermal advection occurs makes all the difference (the $\frac{\partial}{\partial p}$ is key)

- typically, thermal advection is strongest at low levels
- note that term B (differential thermal advection) can strengthen/weaken troughs and ridges (unlike vorticity adv. term)

Example:



· low level WAA is occurring in crest of midlevel ridge
· low level CAA is occurring in base of midlevel trough

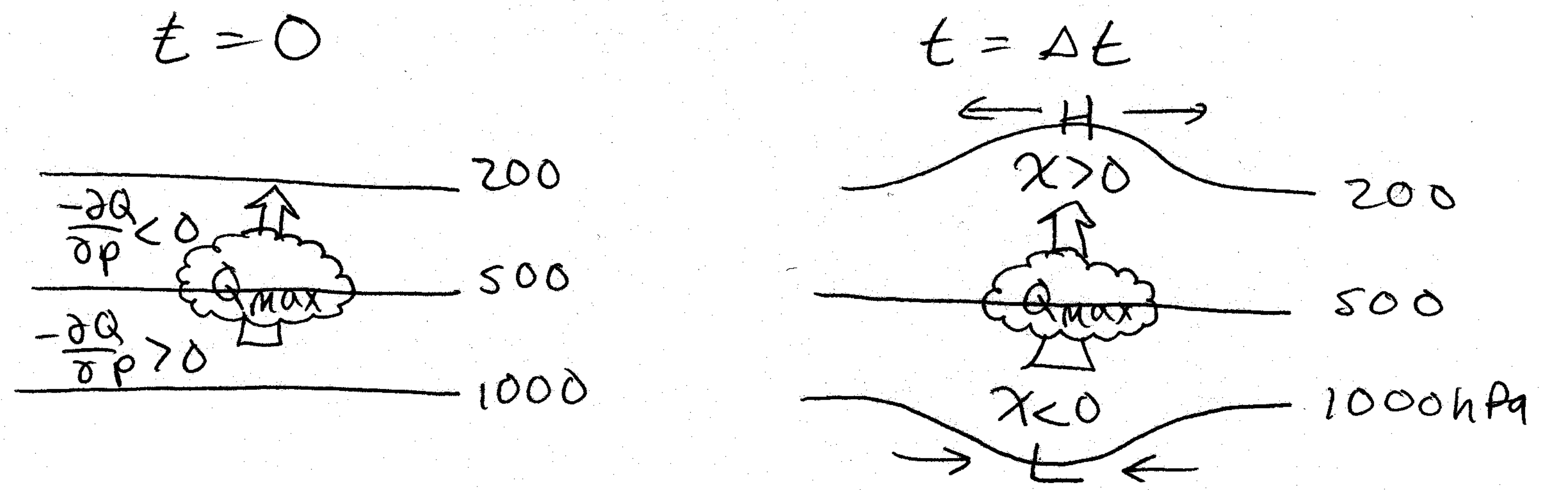
→ 500-hPa pattern will amplify!
(have mutual amplification of surface cyclone and mid-level flow pattern)

· term C:

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi \propto -\frac{\partial Q}{\partial p}$$

→ differential diabatic heating

Example: Diabatic heating at midlevels due to condensation



term D:

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi \propto f_0 \hat{k}\cdot\nabla\times\vec{F}_r$$

→ friction (important in PBL only)

- where Φ is a min, term increases height
- where Φ is a max, term decreases height

→ friction acts to reduce amplitude of pattern