The QG Omega Eqn

- can obtain a diagnostic eqn for vertical motion from QG thermodynamic & vorticity equations

- first, take ∇_p^2 of thermodynamic eqn:

$$\nabla_p^2 \frac{\partial}{\partial t}\left(-\frac{\partial \Phi}{\partial p}\right) = -\nabla_p^2\left[\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \nabla_p^2(\sigma\omega) + \nabla_p^2\left(\frac{Q\sigma}{c_pT}\right)$$

switch order of differentiation

- replace $\frac{\partial \Phi}{\partial t}$ with χ

assume no x, y variation

$$\rightarrow \boxed{-\nabla_p^2 \frac{\partial \chi}{\partial p} = -\nabla_p^2\left[\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \sigma \nabla_p^2\omega + \nabla_p^2\left(\frac{Q\sigma}{c_pT}\right)} \quad (1)$$

- now, take $\frac{\partial}{\partial p}$ of vorticity eqn:

$$\frac{\partial}{\partial p}\left(\frac{\partial \zeta_g}{\partial t}\right) = -\frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla(\zeta_g + f)\right] + \frac{\partial}{\partial p}\left(f_0^2 \frac{\partial \omega}{\partial p}\right) + \frac{\partial}{\partial p}(\hat{k} \cdot \vec{\nabla} \times \vec{F})$$

- sub in $\zeta_g = \frac{1}{f_0}\nabla_p^2\Phi$, multiply by f_0

$$\rightarrow \boxed{\nabla_p^2 \frac{\partial \chi}{\partial p} = -f_0 \frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right)\right] + f_0 \frac{\partial^2 \omega}{\partial p^2} + f_0 \frac{\partial}{\partial p}(\hat{k} \cdot \vec{\nabla} \times \vec{F})} \quad (2)$$

- now add eqns 1 and 2

$$0 = -f_0 \frac{\partial}{\partial p}\left[\vec{V}_g \cdot \nabla\left(\frac{1}{f}\nabla^2\Phi + f\right)\right] + f_0 \frac{\partial^2 \omega}{\partial p^2} + f_0 \frac{\partial}{\partial p}\hat{k}\cdot\vec{\nabla}\times\vec{F}$$

$$-\nabla_p^2\left[\vec{V}_g \cdot \vec{\nabla}\left(-\frac{\partial \Phi}{\partial p}\right)\right] + \sigma\nabla_p^2\omega + \nabla_p^2\left(\frac{Q\alpha}{c_p T}\right)$$

- bring terms with ω over to LHS
- divide by σ

$$\rightarrow \left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \cdot \nabla\left(\frac{1}{f_0}\nabla^2\Phi + f\right)\right]}_{A} \underbrace{-\frac{1}{\sigma}\nabla^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right]}_{B}$$

$$\underbrace{-\frac{1}{\sigma}\nabla_p^2\left(\frac{Q\alpha}{c_p T}\right)}_{C} \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\hat{k}\cdot\vec{\nabla}\times\vec{F}\right)}_{D}$$

$\rightarrow$ {the QG omega eqn}

Physical Interpretation

LHS: 2nd derivatives of ω in x, y, and p

- exactly the same form as LHS of height tendency eqn

$$\rightarrow \left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\omega$$

- if RHS is positive, ω is negative

$\omega < 0 \rightarrow$ rising motion

Term A: $\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V_g}\cdot\nabla\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right)\right]$

$\rightarrow$ differential vorticity advection

$$-\vec{V_g}\cdot\nabla\left(\underbrace{\frac{1}{f_0}\nabla_p^2\Phi}_{\zeta_g} + f\right) > 0 \rightarrow CVA$$

- if CVA increases with height (the typical situation)

$$\frac{\partial}{\partial p}\left[-\vec{V_g}\cdot\nabla\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right)\right] < 0$$

$$\rightarrow -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V_g}\cdot\nabla\left(\frac{1}{f_0}\nabla_p^2\Phi + f\right)\right] > 0$$

$\rightarrow$ positive forcing $\rightarrow \omega < 0$

- so it's <u>not</u> that <u>CVA</u> causes rising motion, it's that <u>CVA increasing with height</u> (or AVA decreasing with height) <u>forces rising motion</u>

- typically (<u>but not always!</u>), vorticity advection is stronger at mid and upper levels than near the surface

→ usually can get away with looking at midlevel vorticity advection (why we look at 500-hPa maps of height and vorticity!)

- CVA increasing with height
  -or-
- AVA decreasing with height

"QG forcing for ascent"

- CVA decreasing with height
  -or-
- AVA increasing with height

"QG forcing for descent"

Term B: $\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\frac{1}{\sigma}\nabla_p^2\left[-\vec{V}_g \cdot \nabla\left(-\underbrace{\frac{\partial \Phi}{\partial p}}_{\sim T}\right)\right]$ (5)

- horizonal maximum of temperature advection (horiz. laplacian of temp. advection

$-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) > 0 \rightarrow$ WAA

$\nabla_p^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0 \rightarrow$ WAA max

$\rightarrow -\frac{1}{\sigma}\nabla_p^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0$

$\rightarrow$ positive forcing $\rightarrow \omega < 0$

$-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right) < 0 \rightarrow$ CAA

$\nabla_p^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] > 0 \rightarrow$ CAA max

$\rightarrow -\frac{1}{\sigma}\nabla_p^2\left[-\vec{V}_g \cdot \nabla\left(-\frac{\partial \Phi}{\partial p}\right)\right] < 0$

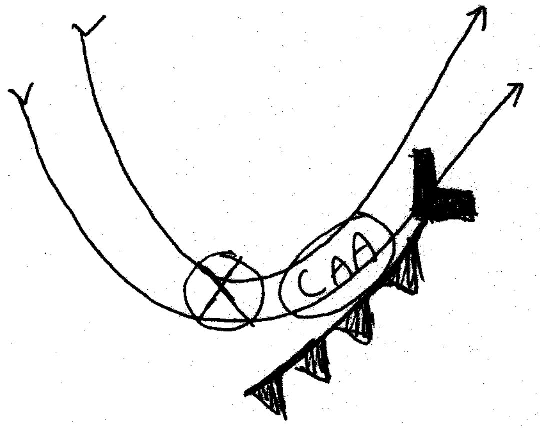
$\rightarrow$ negative forcing $\rightarrow \omega > 0$

- <u>WAA</u> max forces <u>rising motion</u>
- <u>CAA</u> max forces <u>sinking motion</u>

· NOTE: Terms A and B can oppose each other

· we know that CVA increasing with height provides pos. QG forcing

· can have CAA max in presence of CVA increasing w/ height
(CAA max provides neg. QG forcing)



Ⓧ → vort max
⇉ → 500-mb streamlines
L → surface low

→ a limitation of applying QG ω-eqn quantitatively

· can combine terms A and B and get

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \sim \frac{f_0}{\sigma}\, 2\left(\frac{\partial \vec{V}_g}{\partial p} \cdot \nabla \zeta_g\right)$$

(see Holton / Bluestein for details)

$$\rightarrow \left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \sim \frac{f_0}{\sigma} 2\left(-\vec{V}_T \cdot \nabla \zeta_g\right)$$

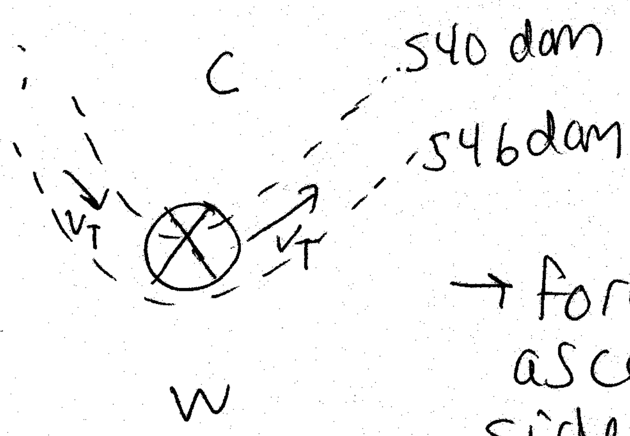
→ advection of vorticity by the thermal wind!

→ called the "Sutcliffe Approximation"

$-\vec{V}_T \cdot \nabla \zeta_g > 0 \rightarrow$ cyclonic vort. advection by thermal wind

$\rightarrow \omega < 0 \rightarrow$ rising motion

example:



⊗ → 700 hPa vort max

- - -→ 1000-500 hPa thickness

→ forcing for ascent on east side of thickness trough

→ forcing for descent on west side of thickness trough

- benefit of plotting 700-hPa relative vorticity and 1000-500 hPa thickness:

→ can eyeball regions of forcing for ascent/descent using <u>one map</u>!
(better than looking at 500-hPa geop. hght, vorticity or 850-hPa isotherms, wind only)

<u>Term C</u>: $\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\frac{1}{\sigma}\nabla_p^2\left(\frac{Q\alpha}{c_pT}\right)$

- diabatic heating/cooling max (laplacian of diabatic heating/cooling)

- very similar to term B, except heating is from diabatic process instead of advection

$\nabla_p^2\left(\frac{Q\alpha}{c_pT}\right) < 0 \rightarrow$ max in diabatic heating

$\rightarrow -\frac{1}{\sigma}\nabla_p^2\left(\frac{Q\alpha}{c_pT}\right) > 0$

$\rightarrow \omega < 0$ (forcing for ascent)

Term D: $\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \propto -\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left(\hat{k}\cdot\vec{\nabla}\times\vec{F}\right)$

- friction impact on ω
- important in PBL only

implication:

<u>cyclonic vorticity</u> in PBL provides forcing for <u>ascent</u>

<u>anticyclonic vorticity</u> in PBL provides forcing for <u>descent</u>

- Terms C and D are often neglected (harder to compute)

<u>QG forcing for vertical motion</u>:
depends on <u>differential vort. advection</u>
-and-
<u>temperature advection (max)</u>

- CVA increasing w/ height, WAA max $\rightarrow \omega < 0$
- AVA increasing w/ height, CAA max $\rightarrow \omega > 0$