Analysis of waves in an anelastic model on the equatorial beta plane and comparison with observations

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Abstract

A non-hydrostatic model on the equatorial beta plane is solved for the most basic solutions of vertically and zonally propagating internal waves. Solutions tilt in the vertical and propagate upward, with buoyancy as a principal restoring force. Results indicate that when frequency is of the same order of magnitude as a reduced buoyancy frequency, the shallow water model equivalent depth depends on frequency. One consequence of this dependence is that Kelvin waves become dispersive at high frequencies. In a complimentary observational analysis, linear regression and a wavenumber frequency spectrum analysis of observed convectively coupled Kelvin and mixed Rossby gravity (MRG) waves are applied to estimate vertical and zonal wavenumbers and frequencies that are consistent with the strongest signals associated with observed convectively coupled waves. Substitution of these parameters into the model yields theoretical structures characterized by vertical and horizontal wavenumbers and frequencies similar to those observed in the middle troposphere. These model solutions demonstrate that the Coriolis terms associated with the horizontal component of the earth’s angular momentum explain zonal phase shifts of roughly 2-9% of the zonal wavelength (at 10°N or S latitude) of observed convectively coupled waves that propagate through environments that are stable to moist deep convection on the zonal scales of the waves.
1. Introduction

Convectively coupled equatorial waves interact with and organize moist deep convection in the tropics (Kiladis et al. 2009). These waves influence tropical cyclogenesis (e.g., Frank and Roundy 2006; Bessafi and Wheeler 2006) and interact with the Madden Julian Oscillation (MJO, e.g., Zhang 2005; Roundy 2008). Many authors have drawn analogs between these waves and zonally propagating wave solutions of the shallow water model on the equatorial beta plane (Matsuno 1966; Lindzen 1967; Kiladis et al. 2009 and references therein). These solutions provide horizontal structures and dispersion characteristics that are remarkably similar to many observed waves, given the crude approximations made, including constant density throughout the fluid and barotropic flow.

In spite of the successes of these shallow water models in predicting the basic patterns of horizontal structure and dispersion, they cannot directly represent some aspects of the observed waves without substantial modification (e.g., Yang et al. 2007 a, b, c; Roundy 2008). Many observed waves would rely on buoyancy as a principal restoring force in an atmosphere statically stable to moist deep convection on the spatial scales of the waves, (e.g., Kiladis et al. 2009 and references therein). Lindzen (1974), Holton (2004), Kiladis et al. (2009), and others have overcome this limitation by modifying the shallow water model to include vertical variation of density. If a rigid lid is not applied at the top of a model troposphere and the atmosphere is assumed to be stably stratified on the spatial scales of the waves, solutions include wave structures that tilt in the vertical, with dispersion characteristics and horizontal structures that depend on vertical wavenumber and the buoyancy frequency (Lindzen 2003). The horizontal structures and dispersion properties of
waves in these models can also be obtained from a shallow water model by choosing a particular equivalent depth.

This work discusses an alternative model framework with which to analyze the three-dimensional structures and dispersion properties of convectively coupled equatorial waves. This model framework simplifies the study of some factors that are difficult to analyze in modified shallow-water models. To illustrate one example application, Kasahara (2003) postulated that the horizontal component of the earth’s angular velocity would be relevant to the dynamics of non-hydrostatic, diabatically driven circulations in the tropics, through the cosine Coriolis terms in the momentum equations (see Gerkema et al. 2005 for a review of these terms). For simplicity, we label these terms the “non traditional Coriolis terms”. These terms couple zonal and vertical flow, and would be balanced in the vertical by buoyancy in an atmosphere statically stable to deep convection on the spatial scales of the waves.

Recent works have demonstrated that these Coriolis terms play important roles in the dynamics of deep atmospheres such as that of Jupiter (e.g., Yano 1994, 1998, 2002, Yano et al., 2003, 2005). Due to a relatively weak density stratification of the deep oceans, others have recently suggested they could be equally important in the equatorial oceans. For example, Fruman et al. (2009) discuss their role in generation of equatorial jet currents. Even in the Earth's atmosphere, which is neither deep nor weakly stratified, the possibility that the non-traditional Coriolis terms play critical roles has not been excluded. This possibility was recently investigated by Kasahara (2003a, b, 2007, 2009). Due to a drastic reduction of effective stratification by diabatic heating associated with moist deep convection and maximization of the cosine of latitude near the equator, the non-traditional Coriolis component might be critically important in the tropical atmosphere.
The present paper addresses this issue under a simple model framework and a comparison with observations. Recently, Fruman (2009) analyzed equatorial wave modes in a simple Boussinesq model on the equatorial beta plane that included these terms. He found that these terms do not influence the dispersion properties of equatorial waves, but that they do influence horizontal structures of waves when the ratio $\Omega/N$ is large (where $\Omega$ is the angular velocity of the earth, and $N$ is the buoyancy frequency). This modification of horizontal structure is expressed as a meridional tilt imposed on the solutions. Fruman’s model is hydrostatic, suggesting that his result is useful only with respect to low frequency motions.

We first discuss a non-hydrostatic model on the equatorial beta plane similar to the Boussinesq model of Kasahara (2003), and the simplest forms of its solutions. We apply this model to generalize the analysis of Fruman (2009) to include high frequency waves, and we briefly summarize the consequences of our solutions for such waves. We further generalize the result of Fruman (2009) to include simple vertical structure. We then apply this simple generalization to analyze a subset of observed synoptic to planetary scale convectively coupled mixed Rossby gravity and Kelvin waves that tilt in the vertical and propagate upward through the troposphere. Finally, we estimate the relevance of the non-traditional Coriolis terms to such observed waves.

2. The Model

We begin with a simple modification of the background density assumption in the Boussinesq model of Kasahara (2003), by replacing reference volume-mean density ($\rho_0$) with $\bar{\rho}(z)$, the horizontal mean density. We assume that density is constant in the horizontal and in time, except in terms multiplied by $g$ (the constant acceleration due to gravity). The
perturbation velocity equations \((u', v', w')\) from a resting basic state are:

\[
\frac{\partial u'}{\partial t} - \beta y v' + 2\Omega w' + \frac{1}{\rho} \frac{\partial p'}{\partial x} = 0, \tag{1}
\]

\[
\frac{\partial v'}{\partial t} + \beta x u' + \frac{1}{\rho} \frac{\partial p'}{\partial y} = 0, \tag{2}
\]

\[
\frac{\partial w'}{\partial t} - 2\Omega u' + \frac{1}{\rho} \frac{\partial p'}{\partial z} = s', \tag{3}
\]

the buoyancy equation is

\[
\frac{\partial s'}{\partial t} + N^2 w' = 0, \tag{4}
\]

and the continuity equation is

\[
\frac{\partial \rho u'}{\partial x} + \frac{\partial \rho v'}{\partial y} + \frac{\partial \rho w'}{\partial z} = 0. \tag{5}
\]

Buoyancy is represented as

\[
s = \frac{g(\rho - \bar{\rho})}{\bar{\rho}}; \tag{6}
\]

the Brunt-Väisälä or buoyancy frequency is

\[
N^2 = \frac{g}{\bar{\theta}} \frac{d\bar{\theta}}{dz}, \tag{7}
\]

where \(\theta\) is potential temperature, \(\Omega\) is the angular velocity of the earth, and \(\beta\) is the meridional gradient of the Coriolis force on the equator. The standard equatorial beta plane approximation for \(\sin(\phi)\) (where \(\phi\) is latitude) is \(\beta y\). We extend the beta plane approximation by including the non-traditional Coriolis terms, approximated by \(2\Omega \cos(\phi) \approx 2\Omega\); this approximation generates an error of less than 14% at +/-30° latitude. \(p'\) represents the non-hydrostatic pressure anomaly (i.e., the full pressure gradient force term in this form represents the part of the pressure gradient force that is not balanced by gravity). Our use of
(3) and (4) allows us to analyze high frequency waves that were excluded from the analysis of Fruman (2009) (See his Section 2) because these equations represent a relaxation of the hydrostatic approximation.

This linearized model is consistent with the anelastic approximation in the vertical (e.g., Ogura and Phillips 1962; Lipps 1990) in that it specifies a reference density that is constant in time but variable in space and excludes sound waves. Traditionally the anelastic approximation has not been applied to the large-scale atmosphere because of the reference density assumption (Vallis, 2006). However, these assumptions are reasonable for the tropical atmosphere because although vertical variations of background density would clearly modulate the flow, horizontal variations of background density are small. Strong fronts do not occur in association with the planetary and synoptic scale convectively coupled waves that are the ultimate focus of this work. This system of equations more completely represents the atmosphere than the Boussinesq and shallow water models.

For simplicity, we assume a background state at rest, although previous works have shown that vertical and meridional shear of the zonal wind (e.g., Xie and Wang 1996) and the vertically averaged zonal wind influence the structures of observed waves.

Multiplication of equations (1-4) through by \( \bar{\rho}(z) \), taking into account that it is a function of \( z \) only generates a new system in terms of momentum instead of velocity:

\[
\frac{\partial \bar{\rho} u'}{\partial t} - \beta y \bar{\rho} v' + 2 \Omega \bar{\rho} w' + \frac{\partial p'}{\partial x} = 0
\]

(8)

\[
\frac{\partial \bar{\rho} v'}{\partial t} + \beta x \bar{\rho} u' \frac{\partial p'}{\partial y} = 0
\]

(9)

\[
\frac{\partial \bar{\rho} w'}{\partial t} - 2 \Omega \bar{\rho} u' \frac{\partial p'}{\partial z} = \bar{\rho} s'.
\]

(10)
\[
\frac{\partial \rho s'}{\partial t} + N^2 \bar{w}' = 0. \tag{11}
\]

Equations (5) and (8)-(11) represent the complete non-hydrostatic system. Since no derivatives in \(z\) occur in equations (1)-(4), and since the form of (5) includes a vertical derivative of \(\bar{w}'\), \(u'\), \(v'\), and \(w'\) will obtainable from solutions for momentum simply by dividing those solutions by the vertical density profile. Thus the only consequence of vertical variation of background density in this model is scaling of the flow and buoyancy in the vertical.

We assume zonally and vertically propagating internal wave solutions of the form

\[
(\bar{p}u', \bar{p}v', \bar{p}w', \bar{p}s') = (\hat{u}, \hat{v}, \hat{w}, \hat{s}) \exp(ikx + l \zeta - \nu t), \tag{12}
\]

where the hatted variables are each functions of \(y\) only. \(k\) and \(l\) are zonal and vertical wavenumbers, respectively; \(\nu\) represents angular frequency. Since the troposphere is not capped to vertical propagation and since we assume that the troposphere is statically stable to deep convection on the spatial scales of the waves, an infinite number of vertical modes would exist (e.g., Lindzen 2003). It is suggested that some linear combination of solutions for different values of \(l\) and \(k\) might yield patterns similar to those in the real atmosphere and would allow the surface boundary condition of zero vertical flow to be met while not requiring zero vertical motion at the tropopause. For more general applications, one would also need to consider vertical variation of \(N\), which could also help to meet the condition of zero vertical flow through the surface (i.e., a solid surface would be consistent with infinite \(N\)). For simplicity, (12) assumes only the most basic solution that can be fit to the observed wave structure of the mid troposphere. Substitution of (12) into equations (5) and (8)-(11) yields the following:

\[
-i \nu \hat{u} - \beta y \hat{v} + 2\Omega \hat{w} + ik \hat{p} = 0 \tag{13}
\]
\[-iv \hat{\nu} + \beta \hat{\nu} + \frac{d \hat{p}}{dy} = 0 \tag{14}\]

\[-iv \hat{\nu} - 2\Omega \hat{\nu} + i\lambda \hat{p} = \hat{s} \tag{15}\]

\[ik \hat{u} + \frac{d \hat{v}}{dy} + i\lambda \hat{w} = 0 \tag{16}\]

\[-iv \hat{s} + N^2 \hat{\nu} = 0 \tag{17}\]

To find a solution to (13)-(17) in terms of \( \hat{\nu} \) analogous to the shallow water model of Matsuno (1966), we begin by solving (17) for \( \hat{s} \), substitute the result into (16)

\[-iv \hat{\nu} - 2\Omega \hat{\nu} + i\lambda \hat{p} = \frac{-iN^2 \hat{\nu}}{v}, \tag{18}\]

and solve (18) for \( \hat{\nu} \)

\[\hat{\nu} = \frac{i}{2\Omega v} \left( - (v^2 - N^2) \hat{\nu} + \nu \lambda \hat{p} \right). \tag{19}\]

Substitute (19) into (13), (14), and (16), solve the resulting (13) for \( \hat{\nu} \), substitute the result into (14) and (16), solve (16) for \( \hat{p} \), and substitute the result into (14) to get an equation in \( \hat{\nu} \) only:

\[\frac{d^2 \hat{\nu}}{dy^2} + \frac{4i\lambda \Omega}{N^2 + 4\Omega^2 - v^2} \frac{d \hat{\nu}}{dy} + \left( i \frac{v^2 - k^2 (N^2 - v^2) - \frac{k\beta}{v} (N^2 - v^2)}{N^2 + 4\Omega^2 - v^2} \right) \hat{\nu} + \frac{2i\Omega \lambda \beta}{N^2 + 4\Omega^2 - v^2} \hat{\nu} - \frac{i\beta^2 v^2}{N^2 + 4\Omega^2 - v^2} \hat{\nu} = 0 \tag{20}\]

Next, apply the transformation

\[\hat{\nu} = \eta(y) \exp \left( - \frac{i}{2} \Gamma y^2 \right), \tag{21}\]

where

\[\Gamma \equiv \frac{2\beta \Omega}{N^2 + 4\Omega^2 - v^2}, \tag{22}\]

This transformation is applied because it eliminates the first order derivative and the imaginary part of the zeroth order derivative; it also modifies the term in \( y^2 \) and generates an
equation in $\eta$ for which there exist familiar solutions. This transformation is similar to that applied by Fruman (2009) and by Gerkema and Shrira (2005). The similar transformation of Fruman (2009) does not depend on $\nu$ as a consequence of his assumption of hydrostatic balance. When considered in the context of (12), the complex exponential portion of (21) results in a latitude-dependent, zonal phase shift $\delta$ in solutions for $v'$, where $\delta$ is the contents of the complex exponential in (21) with the $-i$ omitted (consistent with the phase shift diagnosed by Fruman 2009). After applying the transformation, (20) reduces to

$$
\frac{d^2\eta}{dy^2} - \left( \frac{-l^2v^2 + k^2(N^2 - v^2) + \frac{k\beta}{v}(N^2 - v^2)}{N^2 + 4\Omega^2 - v^2} + \frac{y^2\beta^2 l^2(N^2 - v^2)}{(N^2 + 4\Omega^2 - v^2)^2} \right) \eta = 0 \quad (23)
$$

Equation (23) has the same form as the final equation for $\hat{v}$ of Matsuno (1966), and is known as a Weber equation (Polyanin and Zaitsev 2003). This equation has a broad set of solutions, but one particular class of solutions includes the discrete meridional modes familiar from equatorial beta plane shallow water theory (e.g., Matsuno 1966). This class of solutions occurs only if

$$
\left( \frac{(N^2 - v^2)^{1/2}}{\beta l} \right) \left( -k^2 - \frac{k\beta}{v} - \frac{l^2v^2}{N^2 - v^2} \right) = 2n + 1, \quad (24)
$$

where $n$ is an integer. Equations (23) and (24) generalize the result of Fruman (2009) to include simple vertical structure and a correction in terms of $\nu$ that appears due to relaxation of the hydrostatic approximation in this model. Solutions were found by taking the roots of the quadratic in $k$. Figure 1 shows the associated dispersion diagram giving the solutions for $n=-1$ through $n=1$, all assuming $N=5.1\times10^{-3}\text{s}^{-1}$ and $l^1=2\times10^4\text{m}$. Analogous solutions to those of shallow water theory are labeled on Fig. 1, and include the Kelvin, ER, MRG, EIG, and IG modes. The dashed curve is associated with a negative root of the Kelvin wave, and
It has previously been deemed unphysical because its associated zonal wind anomalies do not taper to zero with latitude, thus making it inconsistent with the equatorial beta plane assumption (Matsuno 1966).

Although the values of $l$ and $N$ applied to obtain Fig. 1 result in dispersion curves that are roughly consistent with the shallow water equivalent depths of observed convectively coupled equatorial waves (e.g., Liebman and Hendon 1990; Wheeler and Kiladis 1999; Roundy and Frank 2004; Yang et al. 2007a,b,c), substantially different values of $N$ and $l$ also yield reasonable dispersion solutions. Comparison of this dispersion relation with the shallow water model of Matsuno (1966) reveals that the shallow water model equivalent depth

$$h_e = \frac{N^2 - \nu^2}{l^2 g}.$$  \hspace{1cm} (25)

This result is equivalent to that of Fruman (2009) except for inclusion of $\nu^2$. This inclusion indicates that relaxation of the hydrostatic approximation yields dependence of $h_e$ on frequency. This dependence would become important when $\nu^2$ is of the same order of magnitude as $N^2$.

If we let

$$\frac{\beta^2 l^2 ((N^2 - \nu^2)}{(N^2 + 4 \Omega^2 - \nu^2)^2} = K^2 > 0,$$  \hspace{1cm} (26)

then the corresponding solutions for the meridional structure of $v$ anomalies are of the form

$$\eta(y) = H_n(\sqrt{K} y) \exp\left(-\frac{K y^2}{2}\right),$$  \hspace{1cm} (27)

where $H_n$ is the $n$’th Hermite polynomial. Therefore, after accounting for the gamma transformation, solutions for $\hat{v}$ are of the form of (21).
Setting *explicit* appearances of $\Omega$ equal to 0 (i.e., retaining only the appearances of $\Omega$ that are implicit in $\beta$) eliminates the contributions of the non-traditional Coriolis terms and reduces the result to the Boussinesq model analog to the shallow water model of Matsuno (1966), except that solutions for wave structure include flow that is weighted in the vertical by density, which causes the strongest wind anomalies to occur at the highest altitudes. This result shows that vertical variation of background density has no other influence on the evolution and structure of the waves in this model.

b. Kelvin Wave

Solutions for the Kelvin wave are found by setting $v' = 0$ everywhere and for all time in equations (5) and (8)-(11), and by solving a selection of four of the resulting equations for $\hat{u}$. Since four equations can be selected from the original five, we found that two forms of the same solution exist in terms of different combinations of parameters. The same gamma transformation applied above for $v'$ also applies to the Kelvin wave solution for $\hat{u}$, and one form of the solutions for the corresponding $\eta$ is identical to (27), except that $u_o$ (the maximum value of the equatorial zonal wind) replaces $H_n$. The dispersion relationship for the Kelvin wave can be found by substituting $-1$ for $n$ into (24). Of the three resulting roots for $v$, the positive root of

$$v^2 = N^2 \frac{k^2}{l^2} \left( \frac{1}{k^2} \frac{k^2}{1 + \frac{k^2}{l^2}} \right)$$

(28)

is the non-hydrostatic analog to the shallow water Kelvin mode. The same result can also be obtained by comparing the combination of parameters in the two solutions for the Kelvin wave noted above. Equation (28) implies that the Kelvin wave is dispersive at wavenumbers
much higher than those plotted in Fig. 1. This dispersion is a consequence of relaxation of the hydrostatic approximation.

c. Reduced Moist Stability

Values for $N$ required to match the dispersion curves to those of observed convectively coupled waves are much smaller than observed values of $N$. Previous works have suggested that the actual value of $N$ felt by a wave is reduced by the influence of the release of latent heat in deep convection coupled to the wave. These works suggest that an effective $N'$ can be reduced from the actual $N$ as

$$N' = \sqrt{1 - \alpha N}$$  \hfill (29)

(e.g., Neelin and Held 1987; Emanuel 1987; Yano and Emanuel 1991; Emanuel et al. 1994; Kiladis et al. 2009). Although some waves might evolve differently, the focus of this work is on waves that propagate in an atmosphere statically stable to moist deep convection on the spatial scales of the waves themselves, with unstable conditions possible on local scales. Such waves would induce wave-scale envelopes of enhanced and suppressed convection without wave-scale vertical overturning circulations. Our use of $\alpha$ under these assumptions is consistent with its application by Emanuel et al. (1994). Although this parameterization is a simplification of the observed system, it has some observational support and allows for simple analytical solutions to the model that might offer insight into the behaviors some observed waves. In order to obtain real-value dispersion parameters from equation (24), it is necessary to assume

$$0 < \alpha < 1 - \left( \frac{v}{N} \right)^2 < 1.$$  \hfill (30)

We thus focus our analysis on a particular subset of atmospheric waves for which $\alpha$ is everywhere less than one. Consistent with this view, previous works have suggested that $\alpha$ is
close to, but less than 1 for most convectively coupled waves (e.g., Gill 1982; Kiladis et al. 2009). Observed values of $N$ are substantially higher in the stratosphere than in the troposphere. Parameterized latent heating effectively reduces $N$ in the troposphere through $\alpha$, which further increases this difference. Hereafter, we apply model parameters that are consistent with the troposphere above the boundary layer throughout the model atmosphere, with the understanding that the solutions will not compare well with observed waves approaching and above the tropopause because of substantial vertical variation in both $\alpha$ and $N$. Observed internal waves of the troposphere would propagate across this region, where they would be recharacterized by the more statically stable local background state and the lack of local release of latent heat in convection. This distinction of the class of waves that are the focus of this work is important in light of the suggestion Yano (2007) that $\alpha$ might vary substantially in space and time and occasionally even take on values greater than 1 in association with some waves (leading to imaginary values of $N'$ that are incompatible with this model, because under such conditions, buoyancy would no longer serve as a restoring force).

3. Diagnosing Model Parameters Consistent with Observed Waves

The above generalization of the result of Fruman (2009) to include vertical wavenumber allows us to analyze observational data in the context of (12), (21), (24), and (29) to estimate the values of $\alpha$ in the mid troposphere associated with a particular subset of observed convectively coupled waves. These parameters also allow us to estimate the zonal phase shifts attributable to the non-traditional Coriolis terms, for observed waves that evolve through the mid troposphere in a manner similar to the model solutions. Wavenumber-frequency spectrum analysis and simple linear regression models based on data filtered for
specific regions of the zonal wavenumber frequency domain are useful for diagnosing the horizontal and vertical structures and dispersion characteristics of coherent convectively coupled equatorial waves (e.g., Wheeler et al. 2000). Model parameters that are consistent with observed equatorial waves are either estimated from the spectrum (e.g., Wheeler and Kiladis 1999, hereafter WK99; Roundy and Frank 2004) or measured from regressed waves (Wheeler et al. 2000).

a. Data

Daily-interpolated outgoing longwave radiation (OLR, Liebman and Smith 1995) data are applied as proxy for moist deep convection. Daily mean zonal and meridional wind, temperature, specific humidity, and geopotential height data are obtained for standard pressure levels from the NCEP/NCAR reanalysis (e.g., Kalnay et al. 1996). The mean and first four harmonics of the seasonal cycle are subtracted from the OLR and wind data to generate anomalies. All data are analyzed for the period June 1, 1974 through December 2007, except that OLR data are missing during 17 March through 31 December 1978.

b. Filtering in the Wavenumber-Frequency Domain

We apply filtering in the zonal wavenumber-frequency domain to isolate signals associated with Kelvin and MRG waves, similar to the approach of WK99 and Roundy and Frank (2004). This filtering is applied by calculating the Fourier transform of a longitude-time array in the zonal direction followed by a similar transform in time. Resulting Fourier coefficients that are outside of a selected wavenumber frequency band are then set to zero. Application of the inverse transform yields the filtered data. The wavenumber frequency bands are selected by enclosing specific broad peaks projecting above the background in the OLR spectrum (WK99), and the filters range in frequency between equivalent depths of 8
and 90m in the shallow water model (as done by WK99). Figure 2 shows the MRG and Kelvin dispersion lines from the shallow water model and the corresponding filter bands superimposed on an OLR spectrum (normalized by dividing by the total spectrum smoothed by 40 applications of a 1-2-1 filter, similar to WK99, Roundy and Frank 2004, and Roundy 2008). The Kelvin band is modified from that of WK99 to exclude some of the low frequency range within the spectral gap apparent between the peaks of the MJO and Kelvin waves (Straub and Kiladis 2003).

Data filtered for the entire bands highlighted in Fig. 2 include signals over a broad range of wavenumbers. Regression models or composites based on such filtered data would include a conglomeration of structures associated with waves across a broad range of spatial scales such that they cannot be easily compared with theoretical waves. In addition, simultaneous inclusion of all wavenumbers would mask any differences between the vertical structures associated with waves characterized by different wavenumbers. For instance, such a broad band could not resolve whether prevalence of individual vertical baroclinic modes or tilted, vertically propagating waves depends on wavenumber. In contrast, individual solutions to the anelastic model represent internal wave structures that are characterized by specific zonal wavenumbers.

In order to diagnose patterns associated with observed waves characterized by specific wavenumbers for easy comparison with the anelastic theory, it is useful to filter within the wave bands for signals corresponding to specific wavenumbers. The resulting filter bands are small (varying in frequency only), and would result in perfect Gibbs ringing in longitude. Such ringing would generate spurious anomalies in the filtered data that would repeat as perfect sinusoids around the globe. However, the analysis below suggests that such filtered
data are effective when applied as bases for regression models constructed to predict patterns in local unfiltered data (e.g., Wheeler et al. 2000). Such filtered data are useful in this context because the regression models would reveal far field structures only where they actually occur locally in the unfiltered anomalies. Regression models based on these filtered data would also reveal phase relationships between different quantities associated with wave motion at the selected zonal scale.

c. Linear Regression Model

Simple linear regression is applied to diagnose coherent structures that tend to be associated with observed wave signals (e.g., Hendon and Salby 1994; Wheeler et al. 2000). Using data filtered for the wavenumber frequency band of selected wave types at specific wavenumbers, we extract a time series at a geographical point. This time series serves as a predictor in regression models at each grid point to diagnose the associated structures. To illustrate, we model the variable $Y$ at the grid point $S$ as

$$Y_s = PA_s,$$

where $P$ is a matrix whose first column is a list of ones and second column is the time series of filtered OLR data at the selected base point. The base index for the Kelvin wave is made by averaging from 5°S to 5°N at 80°E, where variance in the Kelvin wave band is maximized in a pattern symmetric about the equator (Roundy and Frank 2004). The index for the MRG wave is developed by first subtracting filtered data at each latitude across the equator and then averaging the result from 10°S to 10°N; this subtraction acts as an antisymmetric cross-equatorial filter, which would enhance the signals of MRG waves in the filtered data relative to noise (e.g., WK99). The base longitude for the MRG index is 170°E, near the maximum in OLR variance in the MRG band (Roundy and Frank 2004). Averaging across 10°N to 10°S
includes the regions of maximum OLR variance in the MRG band (near +/- 7.5° latitude). \( A_s \) is a vector of regression coefficients at the grid point S. After solving for \( A_s \) at each grid point by least squares minimization, (31) is then applied as a scalar equation to diagnose wave structure by substituting a single value for the second column of \( P_s \) that is representative of the active convective phase of a wave located at the base longitude (we set its value at -1 standard deviation). These regression models are applied to create ‘composite’ anomalies of OLR, \( u \) and \( v \) winds, and density across the global tropics at standard pressure levels. Corresponding geopotential heights are calculated by similar regression to the same base indices, and results are used in plotting to simplify comparison with the anelastic solutions, which are expressed in terms of height instead of pressure. Statistical significance of each regression model is assessed based on the correlation coefficient (e.g., Wilks 2005). Since the regression is accomplished in the time domain, some signal from wavenumbers other than the target wavenumber can appear in results. However, we found such contributions to be small in comparison with signals at the target wavenumber.

d. Comparison of Observations with the Anelastic Wave Solutions

We estimated the best fit of the anelastic wave solutions to observed waves by measuring characteristics of regressed waves and by reading peak frequencies at each wavenumber within a wave band from Fig. 2. The zonal wavenumber \( k \) is approximately specified in the regressed waves because the models are based on data filtered for a specific wavenumber. \( l \) is estimated by measuring the vertical slopes in the longitudinal direction of anomalies of the mid troposphere in the equatorial vertical cross-sections of zonal or meridional wind in a regressed wave, then by multiplying the result by the corresponding value of \( k \). \( N \) is calculated from temperature and specific humidity data from the
NCEP/NCAR reanalysis, and is assumed to be everywhere equal to its value averaged over the troposphere from 850 to 200 hPa and from 10°N to 10°S. \( n \) is assigned the values of -1 and 0 for Kelvin and MRG waves, respectively. Given these values of \( k, l, v, n, \) and \( N, \) we find the solution of equation (24) for \( \alpha \) (given 29) that yields a wave structure consistent with the selected value of \( n. \)

After tuning the model to these observed parameters our principal test of the model is to verify whether the mid-level upward motion implied in the solutions is consistent with the locations of the most negative OLR anomalies in the corresponding regressed waves. We further check that model vertical velocities are consistent with regressed vertical velocities in the mid troposphere associated with waves in the NCEP/NCAR reanalysis. Lastly, we estimate the magnitude of the phase shift \( \delta(y) \) at +/- 10° latitude associated with the inclusion of the non-traditional Coriolis terms, assuming the model parameters estimated from the spectral and regression analyses.

4. Results

a. Comparison of Observed and Anelastic Kelvin Waves

The horizontal structures of regressed Kelvin waves are plotted in Fig. 3a-h for \( k=3 \) through 10, respectively. Horizontal wind anomalies are plotted at 850 hPa. Contours represent regressed OLR anomalies (negative anomalies are shaded). These results suggest that negative OLR anomalies associated with Kelvin waves occur from the low-level confluence region westward through the low-level westerly anomalies. Positive OLR anomalies are located over low-level easterly wind anomalies. Thin black contours enclose regions in which the NCEP/NCAR reanalysis 500 hPa vertical wind is upward and significantly different from zero at the 99% level. These contours suggest that upward motion
at 500 hPa occurs above the confluence point of 850 hPa zonal winds and east of the negative OLR anomalies.

Figure 4 shows the corresponding regressed longitude-height cross-sections for equatorial zonal wind, with eastward flow shaded. Anomalies tilt westward with height through the troposphere, and eastward with height through the stratosphere. Kelvin waves of the lowest wavenumbers are associated with opposite signed anomalies between the lower and upper troposphere, more consistent with the first baroclinic mode through the troposphere (e.g., wavenumbers 1-3, with wavenumbers 1 and 2 not shown). The above internal wave solutions cannot be applied directly to assess the relevance of the non-traditional Coriolis terms to such waves.

For more detailed analysis of wave structure, Fig. 5a shows equatorial u (shading) and density (contours) for the wavenumber 4 Kelvin wave. Figure 5b replaces density contours with reanalysis vertical velocity. Negative density anomalies lead westerly wind anomalies by roughly 90° of phase. Consistent with the above interpretation of regressed OLR anomalies, upward motion occurs roughly collocated with westward flow at each altitude in the Kelvin wave. These results suggest that upward motion in convectively coupled waves lags negative buoyancy anomalies by roughly 90° of phase.

Table I shows parameter values estimated from regressed Kelvin waves and the OLR spectrum in the Kelvin band. Values from Table I are substituted into the solutions for the anelastic Kelvin wave to get results plotted in Fig. 6. Shading in Fig. 6 represents upward motion at near 500 hPa, for comparison with regressed OLR anomalies and regressed reanalysis 500 hPa vertical wind shown in Fig. 3. Figure 7 shows the longitude-height cross-
section of anelastic solutions for equatorial zonal wind anomalies for comparison with the
regressed patterns of zonal wavenumber 4 shown in Fig. 5.

Figure 7 shows the equatorial zonal wind along with density (panel a) and vertical
wind anomalies (panel b) for the wavenumber 4 Kelvin wave for comparison with the
regressed wave in Fig. 5b. Anomalies of upward motion are roughly collocated with
anomalies of westward flow. Density anomalies are 90° out of phase with zonal wind
anomalies (which are also in phase with pressure anomalies of the same sign in the Kelvin
wave). Negative density anomalies thus lead anomalies of upward motion by 90°. Upward
motion also occurs in regions of anomalously low pressure. Results are generally consistent
with the tilted structures observed in Figs. 4 and 5 above the boundary layer and below the
tropopause. Similarity between the tropospheric portions of the cross sections of the
regressed Kelvin waves and the anelastic solutions for $u$ is by design, since the parameter $l$
was estimated from the regressed waves and $k$ is pre-defined. However, these parameters do
not individually predetermine the meridional width of the solution in Fig. 6, nor do they
individually predetermine the relationships between solutions for $u$, $s$, $p$, and $w$. For the
range of values of $\alpha$ considered here, solutions have upward motion collocated with low
pressure lagging positive buoyancy anomalies by roughly 90°. Solutions in which mid-level
vertical wind anomalies correspond closely with OLR anomalies of opposite sign suggest
that the model waves are good analogs to the observed waves. Regressed reanalysis vertical
wind anomalies support this perspective, although the magnitude of biases in these data is
difficult to assess. Nevertheless, the location of active convection over low-level westerly
wind anomalies in the regressed Kelvin waves is consistent with the location of mid-level
upward motion in anelastic waves, taking into account the observed vertical tilt.
Figures 3 and 4 do not include solutions at $k=1$ or 2 for brevity. Those results exhibit more prominent anomalies of opposite signs between the upper and lower troposphere, suggesting that the first baroclinic mode might be more relevant to Kelvin waves of those planetary scales. Such waves are not characterized by a single value of $l$, so we cannot analyze them in comparison with solutions of the form of (12). However, smaller scale waves of the mid troposphere are apparently well represented by these solutions.

b. Observed and Anelastic MRG Waves

As for Kelvin waves, the horizontal structures of regressed MRG waves are plotted in Fig. 8a-f for $k = 2$ through 7, respectively. These results suggest that convective anomalies associated with MRG waves originate within the eastern sides of the cyclonic portions of the circulations, and extend poleward and eastward across part of the low-level anticyclonic portions of the wave (e.g., Liebmann and Hendon 1990). Positive and negative OLR anomalies form patterns that are antisymmetric about the equator. In Fig. 8, for each wavenumber, circulations on the eastern side of the domain are centered near the equator. However, circulations to the west are centered off the equator to the north, consistent with previous climatologies of MRG waves over the West Pacific (e.g., Kiladis et al. 2009). Figure 9 shows the corresponding composite longitude-height cross-sections for equatorial meridional wind, with northward flow shaded. Anomalies in the eastern portion of the domain tilt eastward with height through the troposphere, become more vertical near the tropopause, and tilt westward with height through the stratosphere. Anomalies across the western portion of the domain show distinctly opposite anomalies between the lower and upper troposphere instead of tilted structures. These vertical dipole patterns are consistent with the first baroclinic vertical mode. It is interesting that these structures appear in the
same regions where circulation centers move off the equator. It is likely that the changes in horizontal and vertical structures as the waves progress westward are associated with amplification of the active convective anomalies located in the northern hemisphere (e.g., Dickinson and Molinari 2002; Frank and Roundy 2006; Kiladis et al. 2009). Such patterns might also suggest that the atmosphere becomes unstable to moist deep convection on the spatial scales of those waves, implying failure of (30).

Lines drawn through the tropospheric anomalies on the east side of the domain in Fig. 9 are used to estimate tilt, which is used to find $l$ as discussed in Section 3d. Table II shows parameter values estimated from regressed waves and the OLR spectrum in the MRG band. Comparison of Tables I and II and Figs. 4 and 9 suggests that MRG waves are characterized by shorter vertical wavelengths than Kelvin waves. Values from Table II are substituted into the anelastic solutions for the MRG wave to get the result plotted in Fig. 10. Only the wavenumber 4 MRG wave is plotted because the results for other wavenumbers are similar, except for slight phase shifts between the vertical velocity and horizontal circulation anomalies that are associated with different tilts measured from Fig. 9. Wind vectors are plotted at 1506 m. Shading represents upward motion at 5854 m (500 hPa assuming the mean stratification of the global tropics from $10^\circ$N to $10^\circ$S). These solutions compare favorably with the similar $w$ anomalies in Fig. 8c. Figure 11 shows the longitude-height cross section of equatorial meridional wind anomalies for comparison with the regressed pattern in Fig. 9c. Some of the similarity between the tropospheric portions of the cross sections of the meridional wind anomalies in regressed MRG waves east of the dateline and the anelastic solutions for meridional wind is by design, since the parameter $l$ was measured from the regressed waves and $k$ is pre-defined. However, these parameters do not individually
predetermine the meridional width of the solution in Fig. 10, nor do they individually
determine the zonal phase relationships between solutions for $u$, $v$, and $w$.
Correspondence between vertical wind anomalies in the solutions and regressed OLR anomalies of opposite sign suggest that the model waves are good analogs to the observed waves east of 170°E. This conclusion is not sensitive to changes in the base longitude. The vertically tilted solutions of the anelastic waves suggest that upward motion begins at lower levels in cyclonic flow anomalies in MRG waves, and this upward motion tilts eastward, reaching a maximum at mid levels over the regions of strongest poleward low level flow. Ascent in the upper levels occurs over low-level anticyclonic flow. The model solutions do not represent the poleward horizontal tilt suggested in OLR anomalies associated with observed MRG waves.

These results suggest that in regions where the observed horizontal structures are most similar to theoretical MRG waves, with circulation centers on the equator, the vertical structures are similar to the anelastic modes. The most negative OLR anomalies occur at roughly the same point in the horizontal wave structure as mid level upward motion in the anelastic MRG wave. The tilted vertical motion in the anelastic solution is consistent with a progression from shallow convection beginning within the low-level cyclonic flow, to deep convection located in the vicinity of low-level poleward flow, to stratiform and cirrus in the vicinity of low level anticyclonic flow. These patterns are consistent with observations of the convectively coupled MRG wave (e.g., Kiladis et al. 2009). Analysis of the MRG waves east of roughly 170°E suggests a similar relationship between density and vertical motion seen in the Kelvin wave, except that the direction of propagation is reversed.
5. Analysis of the Non-Traditional Coriolis Terms

A first glance at equation (24) reveals that this dispersion relationship does not depend on the non-traditional Coriolis terms (consistent with Fruman 2009). Although (24) does not include these terms, the structural solutions in equation (27) do. A tracer of dependence on these terms is the explicit appearance of $\Omega$ (i.e., ignoring appearances of $\Omega$ in $\beta$). $\Omega$ appears in both $\Gamma$ and $K$. Its appearance in $K$ demonstrates that these terms modify the meridional widths of the wave structures. The explicit dependence in $\Gamma$ results in a zonal phase shift in the wave structures that depends on latitude, $l$, $N$, and $\nu$ that is expressed as a westward tilt with distance from the equator (consistent with Fruman 2009). For example, the poleward ends of circulations in MRG waves occur to the west of the zonal center of the disturbances on the equator. Dependence of $\Gamma$ on $\nu$ is a result of relaxation of the hydrostatic approximation and thus was excluded from the result of Fruman (2009). $\Gamma$ varies significantly with $\nu$ only for periods shorter than about 4 days.

Explicit dependence of $K$ on $\Omega$ disappears when $(1 - \alpha)N^2 - \nu^2 >> 4\Omega^2$. By this interpretation, the non-traditional Coriolis terms are irrelevant to dry waves unless the wave frequency is close to the buoyancy frequency. Further, reduced moist stability is required to generate values of $N$ sufficiently small for these Coriolis terms to be relevant to lower frequency convectively coupled waves. When the non-traditional Coriolis terms are relatively small, the solutions suggest that upward motion occurs collocated at a given height with equatorial easterly winds in the Kelvin wave (above low-level westerly anomalies). As these terms become dominant, upward motion migrates westward until it occurs with eastward flow at a given height.
Observed vertical tilts of regressed waves along with the locations of the corresponding spectral peaks in the wavenumber frequency domain suggest values of parameters representative of observed waves that evolve in a manner consistent with our original assumption of static stability on the spatial scales of the waves. These parameters are applied in Tables I and II to diagnose the magnitudes of $\Gamma$ and the zonal phase shift $\delta$ associated with observed waves. Values of $\Gamma$ are of order $10^{-13}\text{m}^2$. Zonal phase shifts associated with such values of gamma would be zero on the equator and would range from roughly 2 to 9% of the zonal wavelengths at $\pm 10^\circ$ of latitude (See Tables I and II). Waves propagating in less statically stable environments (i.e., $\alpha$ approaching 1) would exhibit greater sensitivity to the non-traditional Coriolis terms. These results do not apply to waves that are apparently well characterized by vertical baroclinic mode structures, because such structures cannot be found in this model. These results also apply only to the average wave that is characterized by vertical tilt through the mid troposphere and these results do not reveal the spread of values of $\alpha$ and $\Gamma$ across the full population of waves, suggesting that higher or lower values may be observed in association with individual wave events.

6. **Summary and Conclusions**

A linear non-hydrostatic model on the equatorial beta plane was solved for the simplest form of its solutions for zonally and vertically propagating internal waves. We extended the similar analysis of Fruman (2009) by relaxing the hydrostatic approximation and by including a simple vertical structure. The dispersion characteristics of the solutions are identical to shallow water model solutions of Matsuno (1966) (given equation 25), and the horizontal structures are also identical except for modifications caused by inclusion of the non-traditional Coriolis terms associated with the horizontal component of the earth’s
angular momentum (Kasahara 2003). Relaxation of hydrostatic balance in the model reveals that the model quantity identical to equivalent depth in the shallow water model must depend on frequency. Another consequence is that the Kelvin wave becomes dispersive at very high zonal wavenumber. Solutions tilt in the vertical against the direction of zonal propagation. Solutions for vertical structure cannot be compared directly with the shallow water model of Matsuno (1966), but are consistent with solutions of similar models that have been modified to allow density variations in the vertical (e.g., Lindzen 1974; Holton 2004; Kiladis et al. 2009).

Regression models based on data filtered for MRG and Kelvin waves characterized by specific zonal wavenumbers and measurable vertical tilts reveal how the structures of observed waves depend on the zonal scales of the waves. The vertical wave number (l) and frequency (ν) were estimated from analysis of the regressed waves and from a wavenumber frequency spectrum of OLR anomalies. Some of the effects of the release of latent heat in deep convection are parameterized by α in equation (29), which is estimated from the dispersion relationship (24), given these values of l and ν, and assuming the mean value of the buoyancy frequency (N) observed through the troposphere of the 10°S to 10°N band. We estimated values of α associated with MRG and Kelvin waves that are characterized by different zonal scales to be between 0.75 and 0.97. Waves characterized by higher zonal wavenumbers appear to be associated with higher values of α and thus greater sensitivity to the non-traditional Coriolis terms (Tables I and II).

Recently, Mapes et al. (2006) considered why patterns in cloudiness associated with large-scale waves evolve in a similar way to mesoscale convective systems, with shallow convection developing first, followed by deep convection and then by upper level stratus.
They suggested that the waves are associated with patterns of anomalous density (controlled by temperature and humidity) that tilt in the vertical in a manner that modifies the life cycles of mesoscale convective systems embedded within. According to their view, such convective systems would spend a greater amount of time in the stages of their life cycles that are consistent with the local large-scale environment set by the wave (i.e., the deep convective phases of mesoscale systems would last longer when the systems are embedded within the portion of the wave characterized by the strongest upward motion through the mid troposphere). Although the anelastic solutions do not shed light on the specific temporal lifecycles of convective systems embedded within, the association of anomalies of non-hydrostatic pressure perturbations, buoyancy, and vertical motion on the spatial scales of the waves would favor a general progression of convection across the wave structure from shallow to deep convective to upper level stratus in spite of the scales over which the convective elements are organized within the wave (i.e., both meso and microscale elements would respond to patterns of vertical motion tied to the large-scale wave).

The anelastic theory supports this perspective because upward motion associated with observed waves follows roughly 90° of phase behind positive buoyancy anomalies that tilt in the vertical against the direction of wave propagation. Anomalous downward flow begins to accelerate upward in response to the advance of regions of positive buoyancy, and the first upward motion at a given longitude occurs in the low levels, with subsidence remaining above. Such broad-scale low-level upward motion capped by subsidence favors development of a population of convective systems dominated by shallow convection. As the anelastic wave progresses, the buoyancy anomalies extend upward through the mid-levels, after which time model upward motion favors a population of convective systems characterized by
enhanced moist deep convection. Later, negative density anomalies occur in the upper
troposphere with positive density anomalies beneath, consistent with the observed adjustment
toward stratiform rain. The anelastic theory thus supports the perspective that the large-scale
wave conditions the environment for development of cloudiness in a progression from
shallow to deep to stratiform. Convection modifies the background stability through the $\alpha$
term such that the waves and convection depend on each other for their overall behaviors.
The phase angle between anomalies of buoyancy and vertical motion is modified by the non-
traditional Coriolis terms.

For $\alpha$ greater than 0.7 and less than about 0.95, the impacts of these terms would
account for a few percent of the wavelength—such that their impacts would likely be
statistically indistinguishable from noise. Nevertheless, numerical weather prediction models
including these terms might better resolve the structures of such waves. As $\alpha$ approaches 1,
the impacts of these terms would become more substantial. Relaxation of our assumptions of
constant $\alpha$ and $N$ would contribute further structure to the patterns seen in Figs. 4-5 and 9-10,
but would make analytical solutions more difficult or impossible.

Solutions of the equatorial beta plane shallow water model are frequently applied to
help students interpret the structures and propagation characteristics of convectively coupled
equatorial waves. In light of the anelastic model solutions and previous solutions of other
models that also depend on buoyancy as a restoring force, it is important to distinguish
observed waves of the tropical atmosphere from their shallow water counterparts. For
example, shallow water models and anelastic models induce vertical motion in different
ways. To illustrate, consider the MRG wave. It is interesting that both shallow water and
anelastic models induce upward motion over low-level poleward flow in the MRG wave. At
a given height, upward motion in observed and anelastic MRG waves with $\alpha$ less than about 0.95 occurs with cyclonic flow at any height. Since the region of anomalously low density tilts eastward with height, the maximum upward motion at mid levels occurs roughly over the low-level poleward flow. Upward motion in the shallow water MRG wave occurs together with deep confluence of the wind. Variation of density implies that horizontal convergence does not need to be associated with upward motion for mass conservation, as would be required in a shallow water model. It is thus apparent that the shallow water model leads to the right result for the MRG wave for the wrong reason, since observations suggest that buoyancy associated with density variations that are continuous in the horizontal is a leading restoring force for the waves. Anelastic Kelvin waves also exhibit buoyancy-induced upward motion collocated with negative non-hydrostatic pressure perturbations at a given height such that upward motion occurs together with easterly winds rather than confluence of the winds, and westward vertical tilt would lead to maximum mid-level upward motion consistent with organized convection over surface westerly anomalies. Our results thus support the previously suggested notion that density variations that are continuous in the horizontal are required to correctly describe the vertical velocity fields in observed convectively coupled waves that propagate upward through the troposphere (e.g., Holton 2004; Kiladis et al. 2009).

Some caveats must be considered along with the above conclusions. Only internal waves were analyzed. Although such waves seem to match well the characteristics of many observed waves of the mid troposphere (as suggested by the regression analysis), boundary conditions were not considered, and would surely be relevant to the details of the flow, especially within the boundary layer. No attempt was made to analyze the effects of friction
in the boundary layer or the influence of changes in effective stability near and above the tropopause. These anelastic model solutions do not explain wave behavior when structures associated with the first baroclinic vertical mode become more prominent than structures that tilt in the vertical (such as for planetary scale Kelvin waves and for waves in the MRG band west of about 170°E). Density variations were considered without distinguishing between the separate contributions of temperature and humidity. Further, the vertical wavenumber values were estimated from regression analyses and were simply substituted into the anelastic solutions. Our results therefore raise questions. For example, what processes determine the vertical wavenumber of observed waves? Further, a broad range of estimates for the values of $\alpha$ and $\delta$ suggest the possibility that waves of different spatial scales might be characterized by different efficiencies with which moist convection reduces the cooling rate associated with moist ascent. The results do not offer any explanation for how such scale-dependent differences might occur. In addition, although wave theories have gained broad support in the community, distinctions between observed and theoretical waves have led some to consider that some observed disturbances might evolve as balanced phenomena rather than waves (e.g., Yano and Bonazzola 2009, Yano et al. 2009, Delayen and Yano 2009). Buoyancy would not act as a restoring force in such disturbances. The anelastic model would only be applicable to observed disturbances for which buoyancy is a principal restoring force. Our regression results suggest that a subset of observed waves evolve in a manner consistent with buoyancy as a leading restoring force, but the anelastic model cannot be applied to evaluate disturbances that evolve differently.

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Works Cited


Fruman, M. D., B. L. Hua, and R. Schopp, 2009: Equatorial zonal jet formation through the barotropic instability of low-frequency mixed Rossby-gravity waves, equilibrated by inertial instability, and transition to superrotation. *J. Atmos. Sci.*, 66, 2600-2619.


List of Tables:

Table I: Parameter values for MRG waves characterized by specific wavenumbers. The vertical wavelength and peak period were measured from regressed waves and from the spectrum (respectively), whereas the bottom 3 quantities were estimated from the anelastic model dispersion relationship and from wave solutions.

Table II: Same as Table I, except for parameter values associated with convectively coupled Kelvin waves.
Table I: Parameter values for Kelvin waves characterized by specific wavenumbers.

The vertical wavelength and peak period were measured from regressed waves and from the spectrum (respectively), whereas the bottom 3 quantities were estimated from the anelastic model dispersion relationship and from wave solutions.
Table II: MRG Wave

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<th>4</th>
<th>5</th>
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<td>5%</td>
<td>6.1%</td>
<td>7.2%</td>
<td>9.4%</td>
</tr>
</tbody>
</table>

Table II: Same as Table I except for the MRG wave.
List of Figures

Figure 1: Dispersion curves in the zonal wavenumber frequency domain for solutions to equation (24), as expressed in equation (24), characterized by the values of $N$ and vertical wavelength given in the figure title. The dashed solution has been deemed unphysical, consistent with Matsuno (1966). Zonal wavenumber is the circumference of the earth divided by the zonal wavelength.

Figure 2  Wavenumber-frequency spectrum of OLR anomalies from 15°N to 15°S. Results are normalized by dividing by a smoothed red background (see text). Dark shades represent power above the background. Dispersion curves from equatorial beta plane shallow water theory are included for reference, for equivalent depths of 8 and 90m. Filters for the Kelvin and MRG bands are labeled.

Figure 3  OLR and 850 hPa wind anomalies regressed against Kelvin band OLR filtered for the wavenumbers noted in the panel titles. Negative OLR anomalies consistent with enhanced moist deep convection are shaded. OLR anomaly contours are plotted every 1Wm$^{-2}$. Thin black contours enclose regions in which the NCEP/NCAR 500 hPa w wind is upward and significantly different from zero at the 99% level.

Figure 4 Equatorial $u$ wind anomalies regressed against OLR anomalies at 80°E filtered for the wavenumbers indicated in the panel titles within the Kelvin band shown in Fig. 2. Eastward flow anomalies are shaded. The maximum amplitude is about 0.5ms$^{-1}$. Lines drawn on some anomalies were used to estimate vertical slope, and resulted in the estimates for vertical wavenumber written on the diagrams.

Figure 5  a. Regressed zonal wind (shading) and density (contours, with the interval posted on the upper-right) associated with the wavenumber 4 Kelvin wave shown in Fig. 4b.
Positive density anomalies are indicated in dark gray, and the zero contour is omitted.  

b. As in (a), except that regressed NCEP/NCAR reanalysis vertical velocity is contoured instead of density, with upward motion indicated in dark gray. Vertical velocity is contoured roughly every 0.003 ms$^{-1}$, with the zero contour omitted.

Figure 6 $u$ and $v$ winds (vectors, at 1.5km height) and $w$ (shading) at 5854m for the $k=4$ Kelvin wave. $l$ is represented in terms of vertical wavelength. Dark shades represent upward motion.

Figure 7 Wavenumber 4 anelastic Kelvin wave zonal wind on the equator (shading), with contours representing (a) density, and (b) vertical motion, for comparison with Fig. 5. Positive values are contoured in dark gray.

Figure 8 Regressed OLR (shading) and vectors for the horizontal wind (at 850 hPa), based on OLR anomalies filtered for specific wavenumbers within the MRG band. Negative OLR anomalies consistent with enhanced moist deep convection are shaded. Panels a-f show results for zonal wavenumbers 2-7, respectively. The longest wind vector is roughly 0.5 ms$^{-1}$. Thin black contours enclose regions where the regressed NCEP/NCAR reanalysis $w$ wind at 500 hPa is upward and significantly different from zero at the 99% level.

Figure 9 Regressed equatorial $v$ wind anomalies plotted with respect to height corresponding to the MRG-wave regression results shown in Fig. 8. Lines drawn on some anomalies were used to estimate vertical slope, and resulted in the estimates for vertical wavenumber written on the diagrams. Positive values are shaded, the zero contour is omitted, and the maximum amplitude is close to 0.5 ms$^{-1}$.
Figure 10  $u$ and $v$ winds (vectors, at 1506m height) and $w$ at 5854m for MRG wave solutions at zonal wavenumber 5. $l$ is expressed in terms of vertical wavelength.

Figure 11  $u$ wind (shading), and $w$ wind (contours, dark gray represents upward motion) for the anelastic solutions for the wavenumber 4 MRG wave at 7°N. The contour interval for vertical velocity is $3.4 \times 10^{-4}$ ms$^{-1}$. 
Figure 1: Dispersion curves in the zonal wavenumber frequency domain for solutions to equation (24), as expressed in equation (25), characterized by the values of $N$ and vertical wavelength given in the figure title. Specific wave modes represented by different curves are labeled. The dashed curve has been previously deemed unphysical because the associated zonal wind signal does not vanish with latitude (Matsuno 1966).
Figure 2  Wavenumber-frequency spectrum of OLR anomalies from 15°N to 15°S. Results are normalized by dividing by an estimated red background. Dark shades represent power above the background. Dispersion curves from equatorial beta plane shallow water theory are included for reference, for equivalent depths of 8 and 90m. Filters for the Kelvin and MRG bands are labeled.
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Figure 4 Equatorial $u$ wind anomalies regressed against OLR anomalies at 80°E filtered for the wavenumbers indicated in the panel titles within the Kelvin band shown in Fig. 2. Eastward flow anomalies are shaded. The maximum amplitude is about $0.5\text{ms}^{-1}$. Lines drawn on some anomalies were used to estimate vertical slope, and resulted in the estimates for vertical wavenumber written on the diagrams.
Figure 5  a. Regressed zonal wind (shading) and density (contours, with the interval posted on the upper-right) associated with the wavenumber 4 Kelvin wave shown in Fig. 4b. Positive density anomalies are indicated in dark gray, and the zero contour is omitted. b. As in (a), except that regressed NCEP/NCAR reanalysis vertical velocity is contoured instead of density, with upward motion indicated in dark gray. Vertical velocity is contoured roughly every 0.003 ms\(^{-1}\), with the zero contour omitted.
Figure 6 $u$ and $v$ winds (vectors, at 1.5km height) and $w$ (shading) at 5854m for the $k=4$ Kelvin wave. $l$ is represented in terms of vertical wavelength. Dark shades represent upward motion.
Figure 7  Wavenumber 4 anelastic Kelvin wave zonal wind on the equator (shading), with contours representing (a) density, and (b) vertical motion, for comparison with Fig. 5. Positive values are contoured in dark gray.
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Figure 9  Regressed equatorial v wind anomalies plotted with respect to height corresponding to the MRG-wave regression results shown in Fig. 8. Lines drawn on some anomalies were used to estimate vertical slope, and resulted in the estimates for vertical wavenumber written on the diagrams. Positive values are shaded, the zero contour is omitted, and the maximum amplitude is close to 0.5 ms$^{-1}$. 
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