PROBLEM 1.2.

a. Derive an expression for the decay of the kinetic energy $\frac{3}{2}u^2$ as a function of time.

ANS.

We know that $\quad TKE = \frac{3}{2}u^2 \quad$ ——— (1)

but also that $\quad \frac{d}{dt}(TKE) = -\epsilon \quad$ ——— (2)

and $\quad \epsilon = \dfrac{u^3}{l} \quad$ ——— (3)

hence, from (2):

$$\frac{d}{dt}\left(\frac{3}{2}u^2\right) = -\frac{u^3}{L}$$

$$\Rightarrow \quad \frac{3}{\cancel{2}} \cdot (\cancel{2}u)\frac{du}{dt} = -\frac{u^3}{L}$$

$$\Rightarrow \quad 3u\frac{du}{dt} = -\frac{u^3}{L}$$

$$\Rightarrow \quad \frac{du}{u^2} = -\frac{1}{3L}\,dt$$

then integrating we have:

$$\int_{u(0)}^{u(t)} u^{-2}du = -\frac{1}{3L}\int_0^t dt$$

$$-u^{-1}\Big|_{u(0)}^{u(t)} = -\frac{1}{3L}(t-0)$$

$$-\frac{1}{u}\Big|_{u(0)}^{u(t)} = -\frac{1}{3L}(t)$$

$$-\left(\frac{1}{u(t)} - \frac{1}{u(0)}\right) = -\frac{1}{3L}\,t$$

then

$$\frac{1}{u(t)} - \frac{1}{u(0)} = \frac{t}{3L}$$

$$\Rightarrow \quad u(t) = \frac{1}{\frac{t}{3L} + \frac{1}{u(0)}}$$

$$u(t) = \frac{u(0)}{\frac{u_0 t}{3L} + 1}$$

$$\Rightarrow \quad u(t) = \frac{3L\, u(0)}{3L + u(0)t} \qquad (4)$$

then, the decay of the energy as a function of time will be given by:

$$k(t) = \frac{3}{2}\left(\frac{3L\, u(0)}{3L + u(0)t}\right) \qquad (5)$$

b. Compute c by requiring that the dissipation rate is continuous at $uL/\nu = 10$

ANS.

In this case we know that

$$\epsilon = \frac{c\nu u^2}{L^2} \quad \text{and} \quad \frac{uL}{\nu} = 10$$

If the dissipation rate is continuous, then

$$\frac{u^3}{L} = \frac{c\nu u^2}{L^2}$$

$$\Rightarrow \quad \frac{uL\, u^2}{\nu} = c u^2 = 10$$

$\therefore \quad c = 10.$

PROBLEM 1.3.

To estimate the characteristic velocity $v(r)$ and the characteristic time $t(r)$ of eddies of size r, where r is any length in the range $\eta < r < \ell$ we have that:

$$\epsilon = \frac{u^3}{\ell},$$

hence for $v(r)$ we have

$$\epsilon = \frac{(v(r))^3}{\ell} \Rightarrow (v(r))^3 = \epsilon r \Rightarrow v(r) = (\epsilon r)^{1/3} \quad (1)$$

then for the time we have:

$$t(r) = \frac{r}{v(r)} \Rightarrow t(r) = \frac{r}{(\epsilon r)^{1/3}} = \frac{r}{\epsilon^{1/3} r^{1/3}} = \epsilon^{-1/3} r^{2/3} \therefore$$

$$t(r) = \epsilon^{-1/3} r^{2/3} \quad (2)$$

If this is true, then:

$v(r) = (\epsilon r)^{1/3}$ at $r = \ell$ we have:

$$v(\ell) = (\epsilon \ell)^{1/3} = \left(\frac{u^3}{\ell} \ell\right)^{1/3} = u \quad \checkmark$$

and $$t(\ell) = \epsilon^{-1/3} \ell^{2/3} = \left(\frac{u^3}{\ell}\right)^{-1/3} \ell^{2/3} = u^{-1} \ell \quad \checkmark$$

Finally, if

$$E(k) = k^{-1} v^2(k) \quad ; \quad k = r^{-1}$$

To find an expression for $E(k)$ we have

$$E(k) = \frac{v^2(k)}{k} \qquad \text{———} \qquad (1)$$

We consider now that

$$v^2(k) = v^2\left(\frac{1}{r}\right) = \epsilon^{1/3}\left(\frac{1}{r}\right)^{1/3} = \frac{\epsilon^{1/3}}{r^{1/3}} = \left(\frac{\epsilon}{r}\right)^{1/3} = (\epsilon k)^{1/3}$$

$$v^2(k) = (\epsilon k)^{1/3} \qquad \text{———} \qquad (2)$$

hence substituting (2) in (1) we have:

$$E(k) = \left(\frac{\epsilon k}{k}\right)^{1/3}$$

$$E(k) = \epsilon^{1/3} k^{-2/3} = \frac{\epsilon^{1/3}}{k^{2/3}} = \left(\frac{\epsilon}{k^2}\right)^{1/3}$$

$$\therefore \quad E(k) = \left(\frac{\epsilon}{k^2}\right)^{1/3}$$

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