Inertial Range Properties in 3D and 2D Turbulence

For reference, a few inertial range properties are listed below, omitting non-dimensional constants.

	3D Energy Range	2D Enstrophy Range	
Energy spectrum	$\varepsilon^{2/3}k^{-5/3}$	$\eta^{2/3}k^{-3}$	(T.1)
Turnover time	$\varepsilon^{-1/3}k^{-2/3}$	$\eta^{-1/3}$	(T.2)
Viscous scale, L_ν	$(\nu^3/\varepsilon)^{1/4}$	$(\nu^3/\eta)^{1/6}$	(T.3)
Passive tracer spectrum	$\chi\varepsilon^{-1/3}k^{-5/3}$	$\chi\eta^{-1/3}k^{-1}$	(T.4)

In these expressions:

ν = viscosity, k = wavenumber, ε = energy cascade rate,
η = enstrophy cascade rate, χ = tracer variance cascade rate.

where $\boldsymbol{u} = u\mathbf{i} + v\mathbf{j}$ and $\zeta = \mathbf{k}\cdot\nabla\times\boldsymbol{u}$ and F is a stirring term. In terms of a streamfunction, $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$, and $\zeta = \nabla^2\psi$, and (8.39) may be written as

$$\frac{\partial \nabla^2\psi}{\partial t} + J(\psi, \nabla^2\psi) = F + \nu\nabla^4\psi. \tag{8.40}$$

We obtain an energy equation by multiplying by $-\psi$ and integrating over the domain, and an enstrophy equation by multiplying by ζ and integrating. When $F = \nu = 0$ we find

$$\hat{E} = \frac{1}{2}\int_A (u^2 + v^2)\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla\psi)^2\,\mathrm{d}A, \qquad \frac{\mathrm{d}\hat{E}}{\mathrm{d}t} = 0, \tag{8.41a}$$

$$\hat{Z} = \frac{1}{2}\int_A \zeta^2\,\mathrm{d}A = \frac{1}{2}\int_A (\nabla^2\psi)^2\,\mathrm{d}A, \qquad \frac{\mathrm{d}\hat{Z}}{\mathrm{d}t} = 0, \tag{8.41b}$$

where the integral is over a finite area with either no-normal flow or periodic boundary conditions. The quantity $\hat{E}$ is the energy, and $\hat{Z}$ is known as the *enstrophy*. The enstrophy invariant arises because the vortex stretching term, so important in three-dimensional turbulence, vanishes identically in two dimensions. In fact, because vorticity is conserved on parcels it is clear that the integral of *any* function of vorticity is zero when integrated over A; that is, from (8.39)

$$\frac{\mathrm{D}g(\zeta)}{\mathrm{D}t} = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\int_A g(\zeta)\,\mathrm{d}A = 0. \tag{8.42}$$

where $g(\zeta)$ is an arbitrary function. Of this infinity of conservation properties, enstrophy conservation (with $g(\zeta) = \zeta^2$) in particular has been found to have enormous consequences to the flow of energy between scales, as we will soon discover.[8]