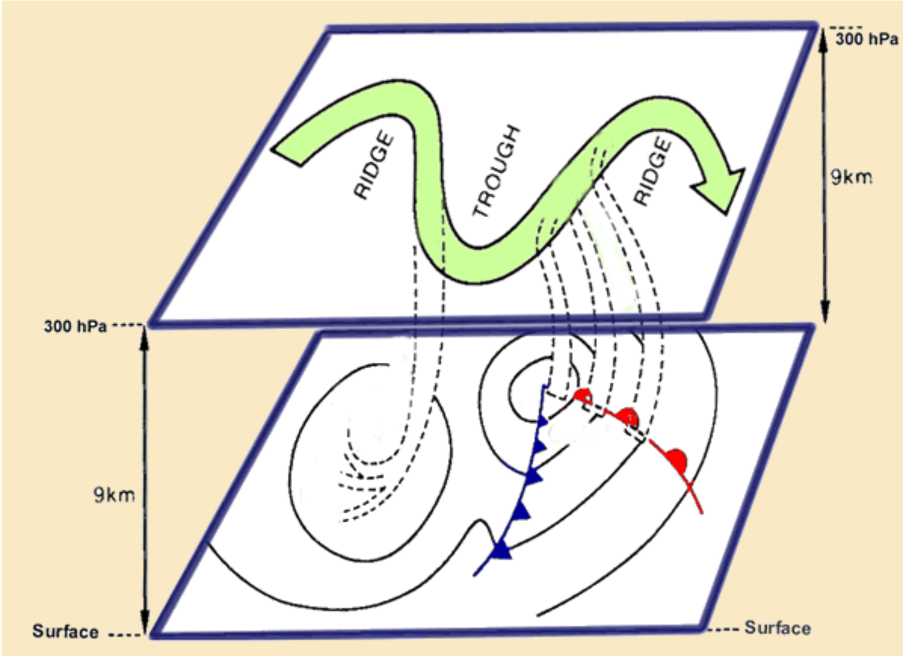
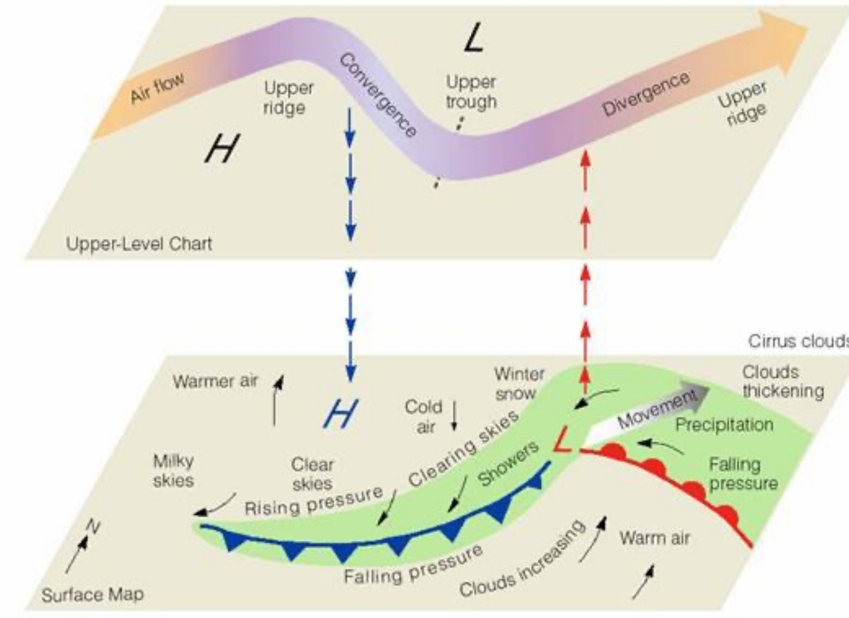
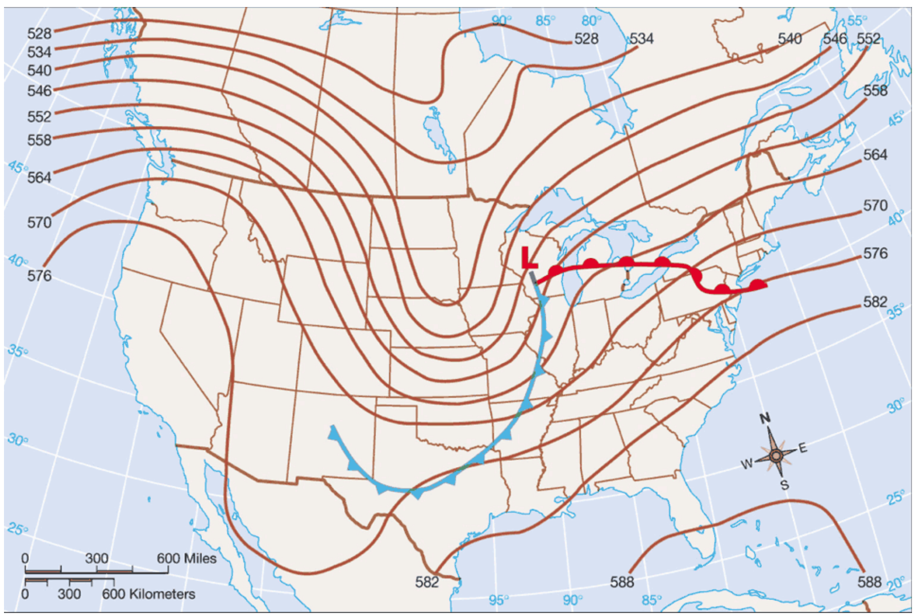
$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2 Q - \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k} \bullet \vec{\nabla}_p \times \overrightarrow{Fr}\right)$$

A B C D E


300 hPa
RIDGE
TROUGH
RIDGE
9km
300 hPa
9km
Surface
Surface


L
Air flow
Upper ridge
Convergence
Upper trough
Divergence
Upper ridge
H
Upper-Level Chart
Cirrus clouds
Warmer air
H
Cold air
Winter snow
Clouds thickening
Movement
Precipitation
L
Milky skies
Clear skies
Clearing skies
Showers
Rising pressure
Falling pressure
N
Falling pressure
Clouds increasing
Warm air
Surface Map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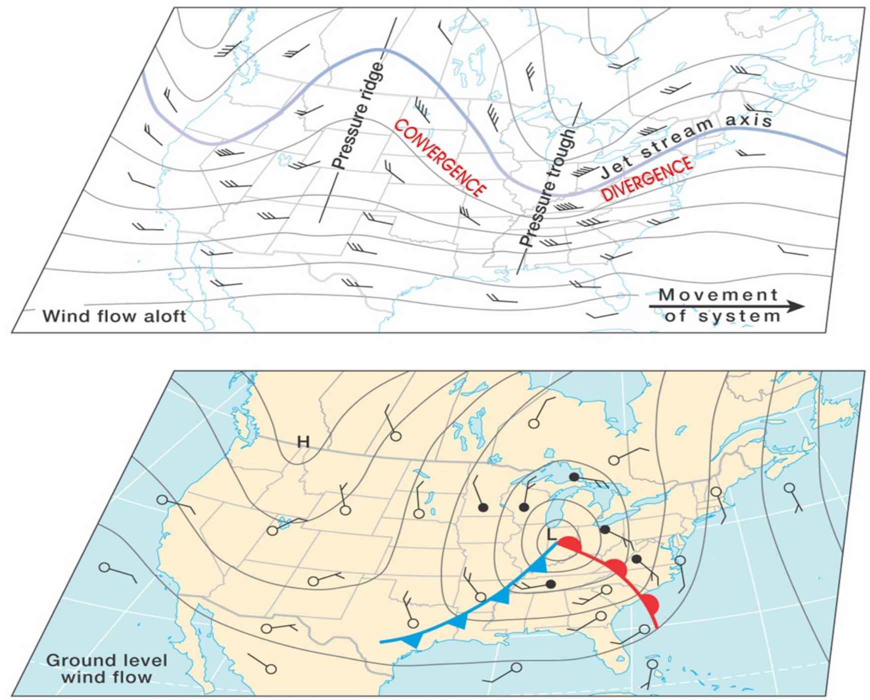
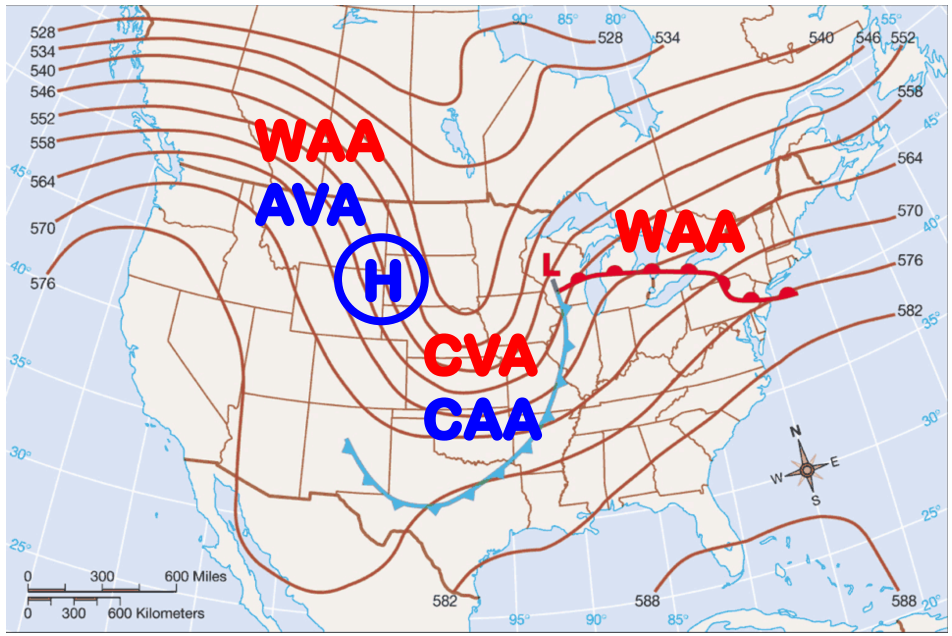

Pressure ridge
CONVERGENCE
Pressure trough
Jet stream axis
DIVERGENCE
Wind flow aloft
Movement of system
H
L
Ground level wind flow

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2 Q - \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k} \bullet \vec{\nabla}_p \times \overrightarrow{Fr}\right)$$

A B C D E

Recalling the typical, synoptic situation:

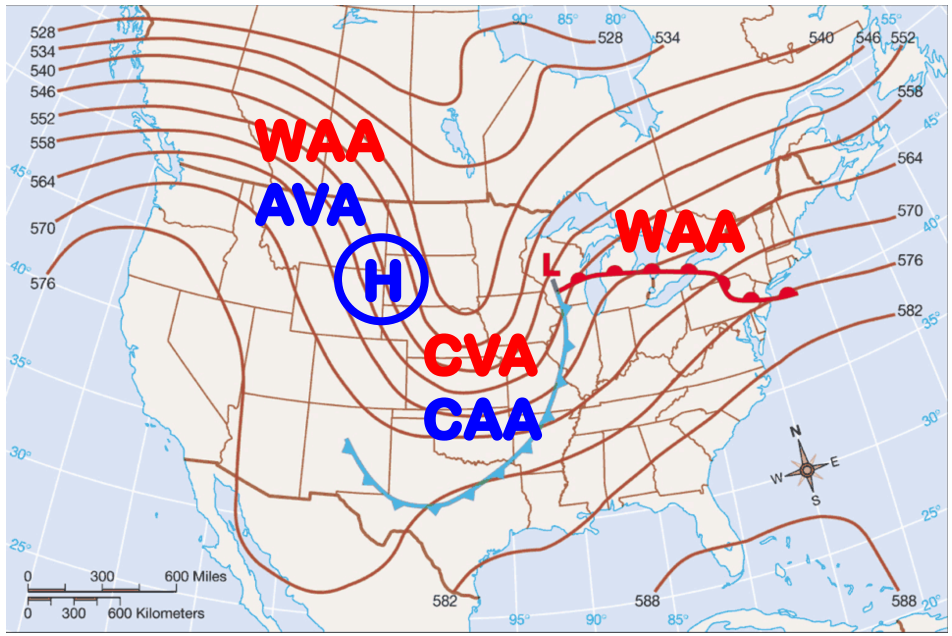


There are regions of significant cancellation between terms B and C, i.e., CVA and CAA behind the cold front, and AVA and WAA to the northwest of the surface high.

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2 Q - \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k} \bullet \vec{\nabla}_p \times \overrightarrow{Fr}\right)$$

A B C D E

Recalling the typical, synoptic situation:

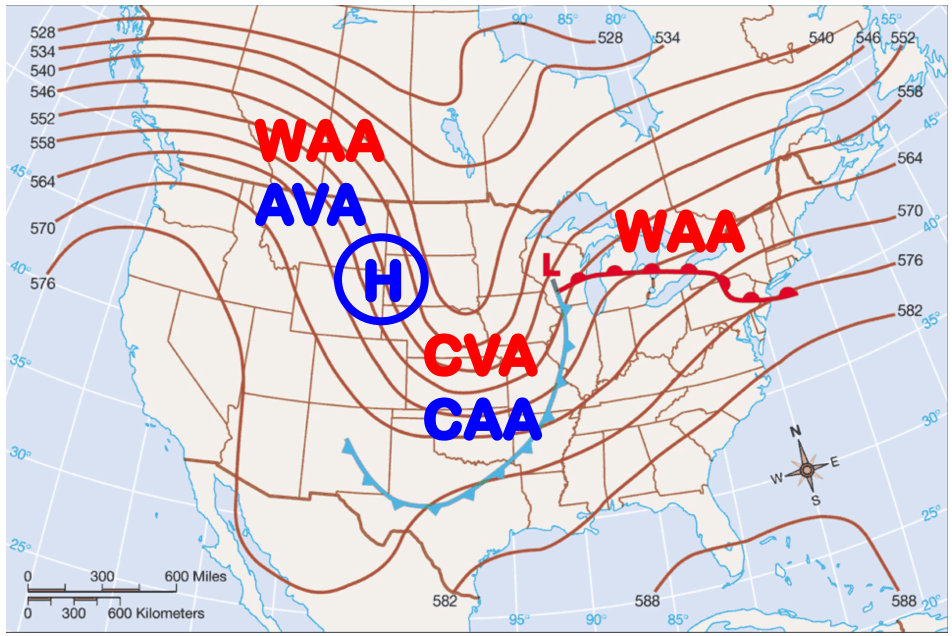


Thus, the QG ω equation can't be used to tell us concretely where upward motion will occur.

$$\left(\vec{\nabla}_p^2+\frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega=-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g\bullet\vec{\nabla}_p\left(\zeta_g+f\right)\right]-\frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g\bullet\vec{\nabla}_p\left(T\right)\right]-\frac{1}{\sigma}\vec{\nabla}_p^2 Q-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k}\bullet\vec{\nabla}_p\times\overrightarrow{Fr}\right)$$

A B C D E

Recalling the typical, synoptic situation:



Sutcliffe (1947) was the first to recognize this, and an alternate form of the ω equation was formally developed by Trenberth (1978), thus the name:

The Sutcliffe–Trenberth ω Equation

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2 Q - \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k} \bullet \vec{\nabla}_p \times \vec{Fr}\right)$$

A B C D E

We start with the adiabatic, frictionless QG ω equation (above) and rewrite term B distributing the – sign and the $\partial/\partial p$:

$$-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] = \frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p\left(\zeta_g + f\right) + \frac{f_o}{\sigma}\vec{V}_g \bullet \frac{\partial}{\partial p}\left[\vec{\nabla}_p\left(\zeta_g + f\right)\right]$$

Next, distribute the $\vec{\nabla}_p$ through the first term and $\partial/\partial p$ through the second:

$$\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p\left(\zeta_g\right) + \frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p f + \frac{f_o}{\sigma}\vec{V}_g \bullet \frac{\partial}{\partial p}\left[\vec{\nabla}_p\left(\zeta_g\right)\right] + \frac{f_o}{\sigma}\vec{V}_g \bullet \frac{\partial}{\partial p}\left(\vec{\nabla}_p f\right)$$

1 2 3 4

We see we can immediately eliminate term 4 because the gradient of the Coriolis force, $\vec{\nabla}_p f$, doesn't change with height.

What about terms 1 and 2? Anything familiar in them?

$$\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p\left(\quad \zeta_g \quad\right)+\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p f+\frac{f_o}{\sigma}\vec{V}_g\bullet\frac{\partial}{\partial p}\left[\vec{\nabla}_p\left(\quad \zeta_g \quad\right)\right]$$

1 **2** **3**

1 = Advection of geostrophic relative vorticity by the thermal wind

2 = Advection of Earth's vorticity by the thermal wind

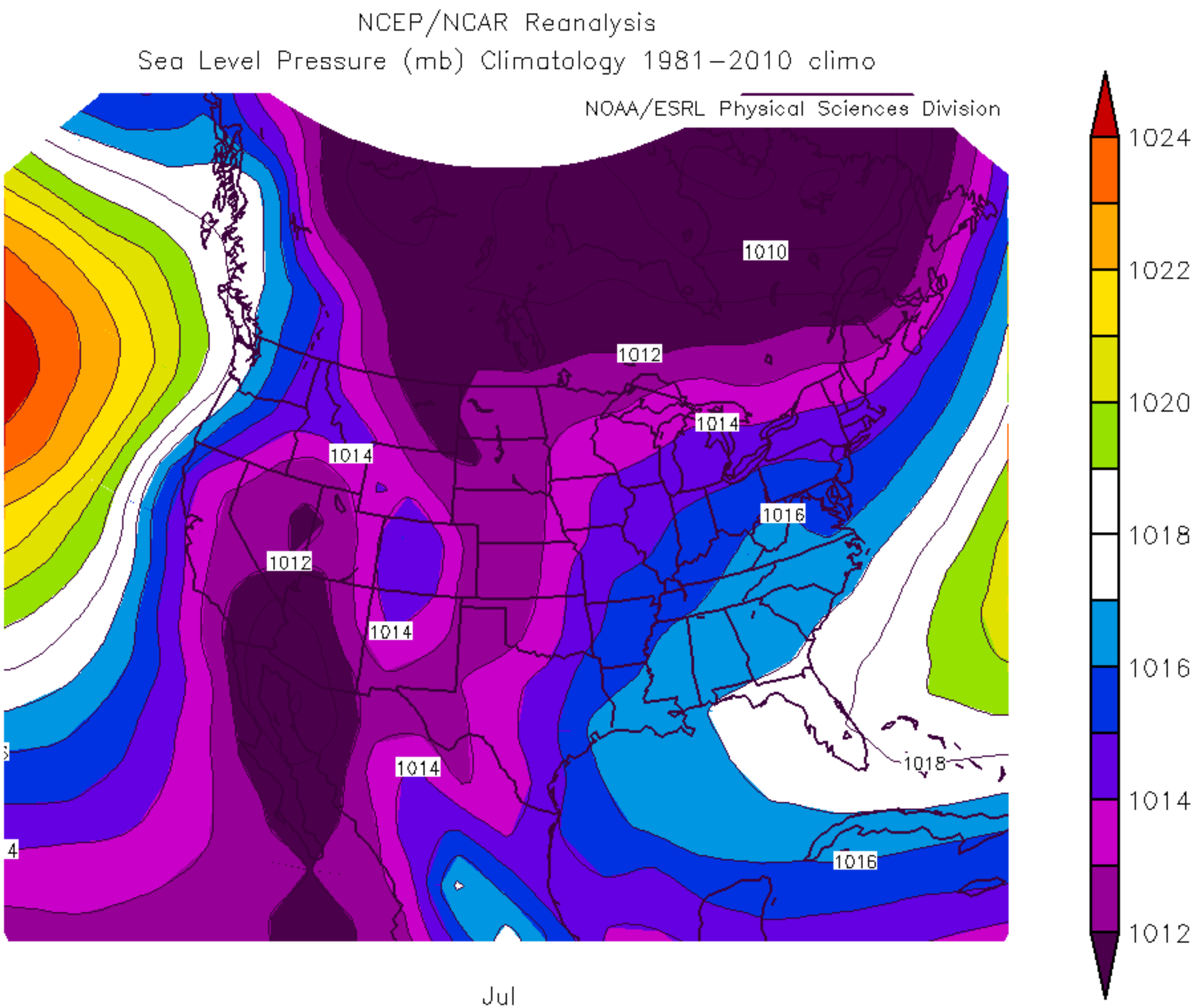
To interpret term 3, let's interchange the order of differentiation (see arrow below) to make it look like an advection term:

$$\frac{f_o}{\sigma}\vec{V}_g\bullet\frac{\partial}{\partial p}\left[\vec{\nabla}_p\left(\quad \zeta_g \quad\right)\right]=\frac{f_o}{\sigma}\vec{V}_g\bullet\vec{\nabla}_p\frac{\partial}{\partial p}\left(\quad \zeta_g \quad\right)=\frac{f_o}{\sigma}\vec{V}_g\bullet\vec{\nabla}_p\frac{\partial \zeta_g}{\partial p}$$

The term is now the advection of the vertical gradient of geostrophic relative vorticity by the geostrophic wind.

It is perfectly fine to leave the term like this, but we give $\frac{\partial \zeta_g}{\partial p}$ a special name…

$$\frac{\partial \vec{V}_g}{\partial p}=\vec{V}_T=\text{Thermal wind} \qquad \frac{\partial \zeta_g}{\partial p}=\zeta_T=\text{Thermal vorticity}$$



NCEP/NCAR Reanalysis
Sea Level Pressure (mb) Climatology 1981–2010 climo
NOAA/ESRL Physical Sciences Division
1010
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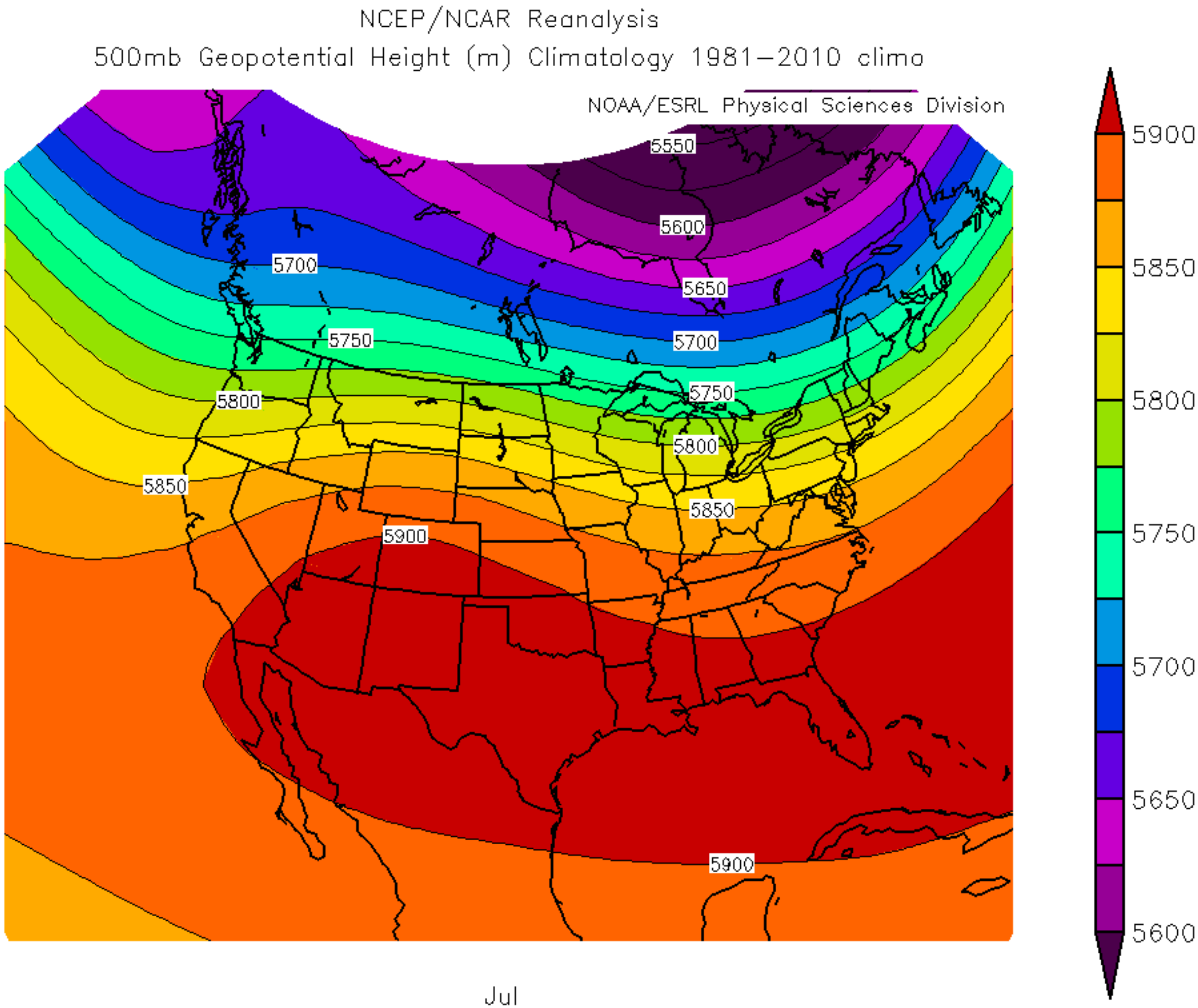
Jul


NCEP/NCAR Reanalysis
500mb Geopotential Height (m) Climatology 1981–2010 climo
NOAA/ESRL Physical Sciences Division
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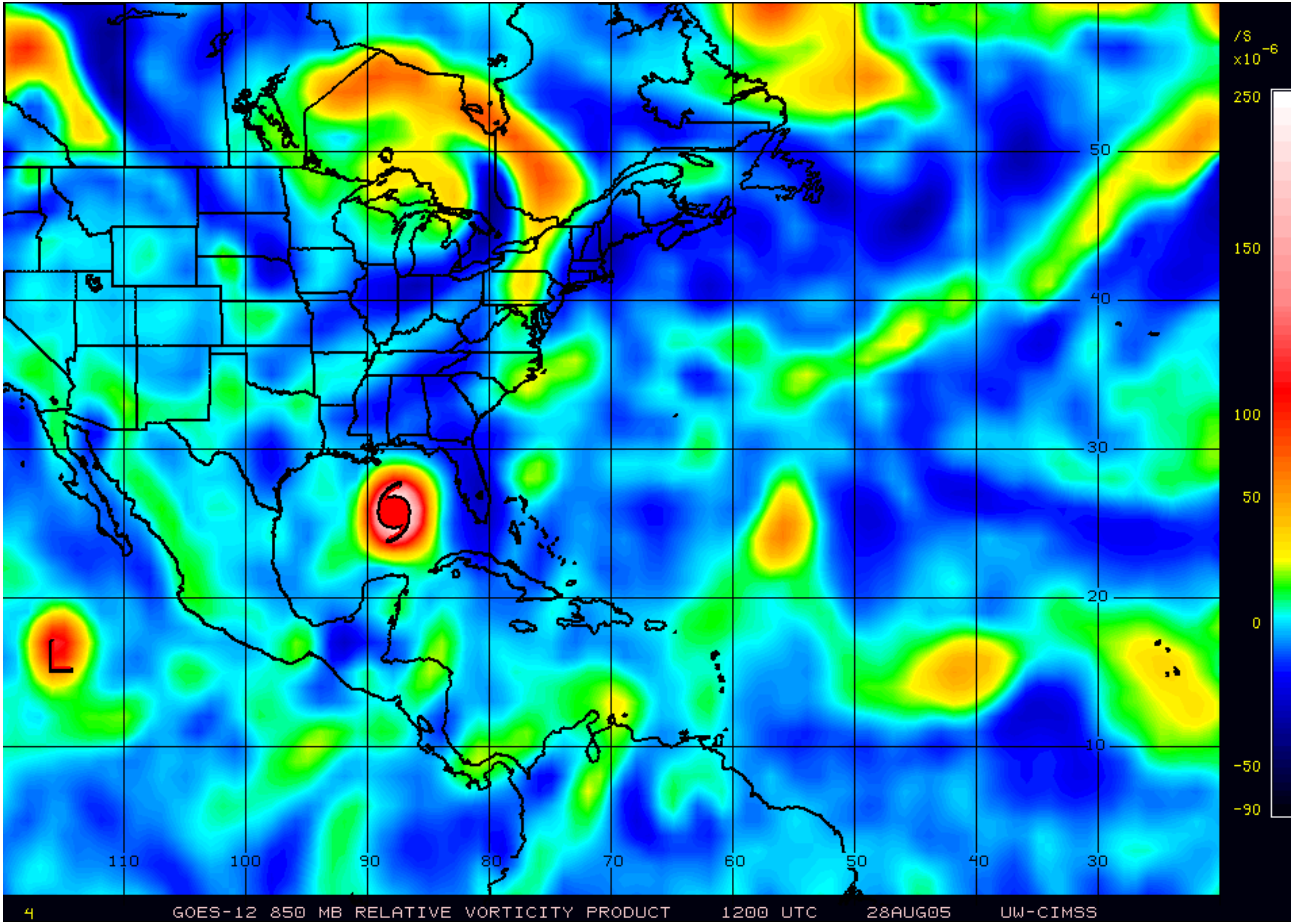
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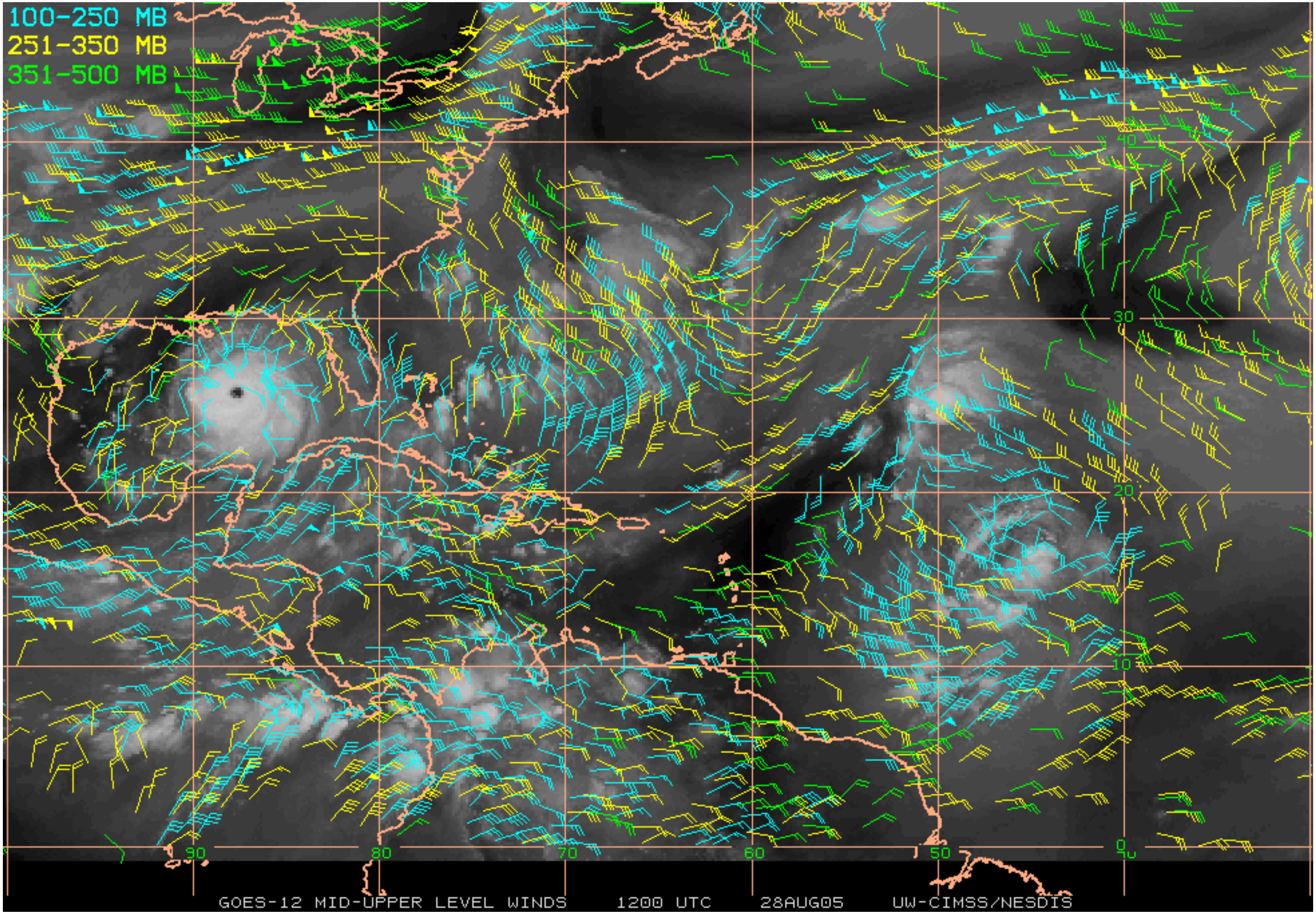
GOES-12 850 MB RELATIVE VORTICITY PRODUCT 1200 UTC 28AUG05 UW-CIMSS


100-250 MB
251-350 MB
351-500 MB
40
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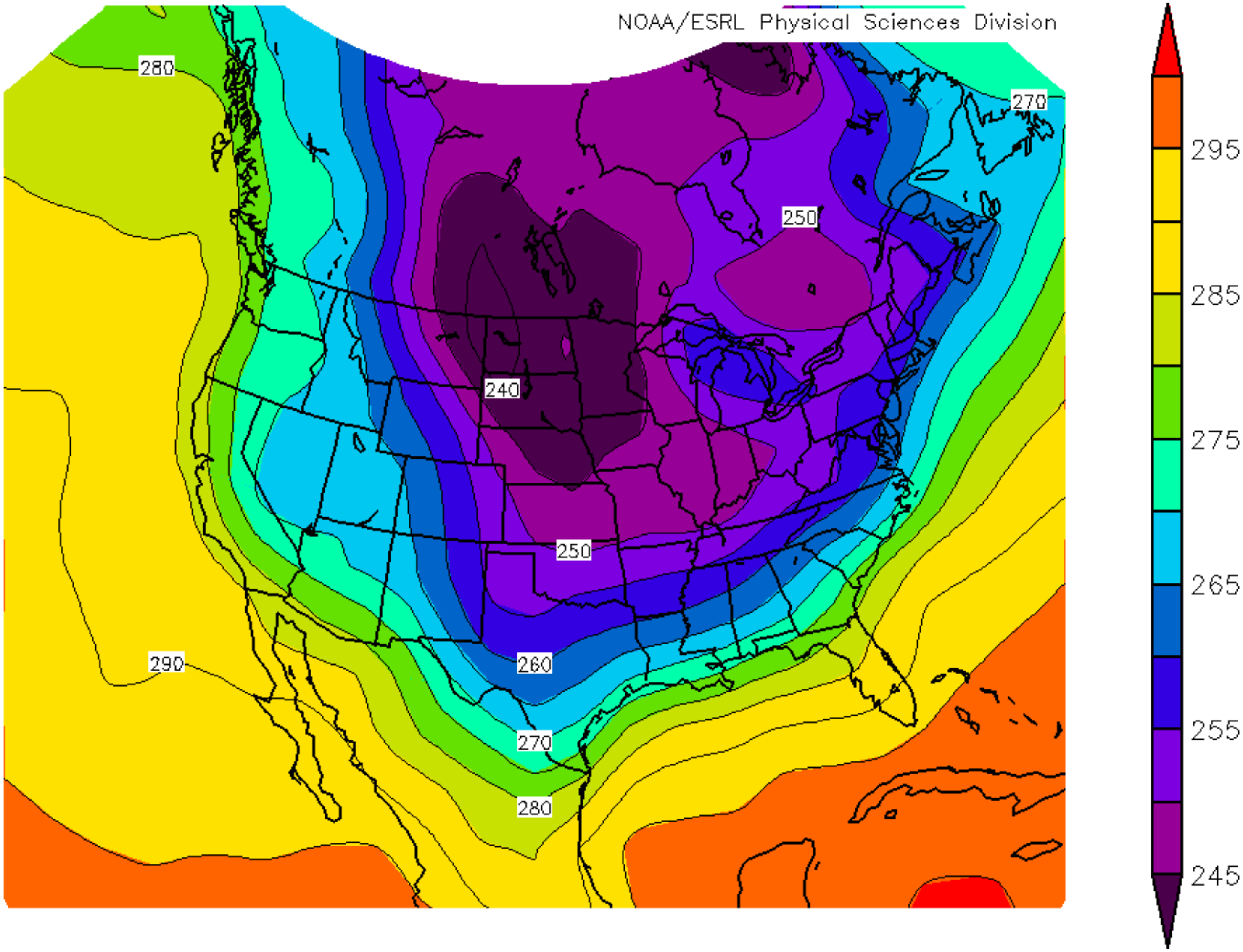
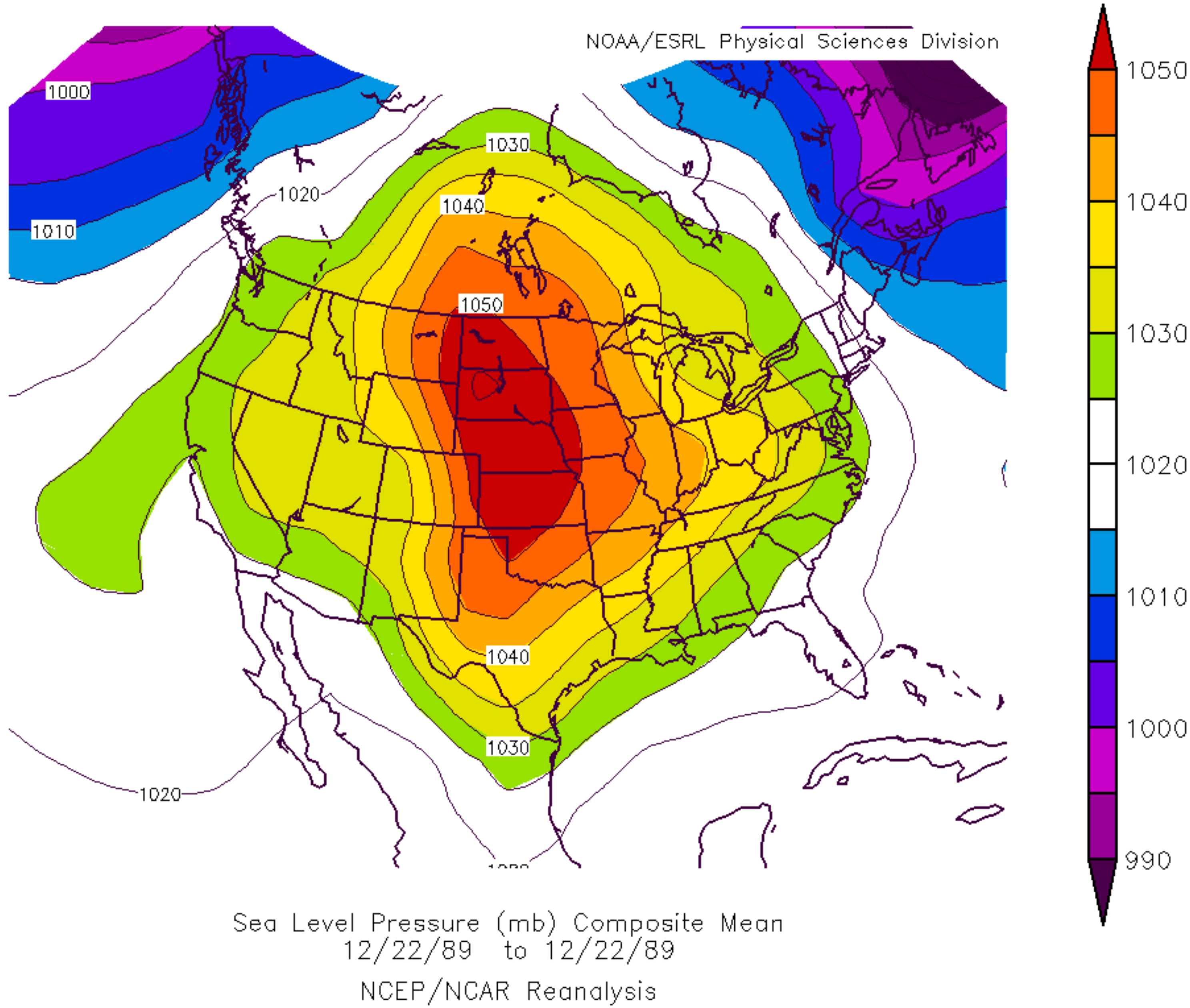
GOES-12 MID-UPPER LEVEL WINDS 1200 UTC 28AUG05 UW-CIMSS/NESDIS



Surface Air Temperature (K) Composite Mean
12/22/89 to 12/22/89
NCEP/NCAR Reanalysis


NOAA/ESRL Physical Sciences Division
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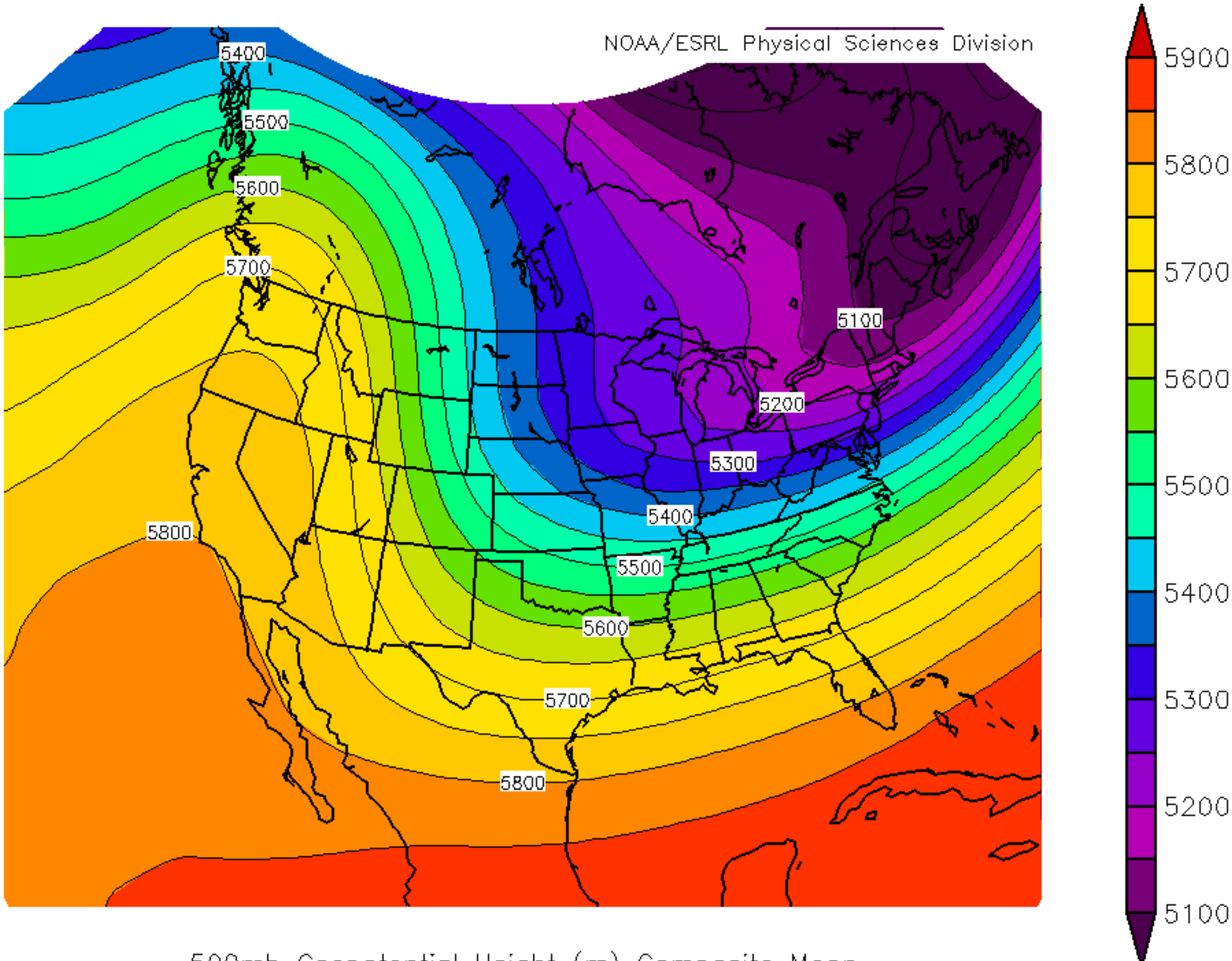
Sea Level Pressure (mb) Composite Mean
12/22/89 to 12/22/89
NCEP/NCAR Reanalysis



500mb Geopotential Height (m) Composite Mean
12/22/89 to 12/22/89
NCEP/NCAR Reanalysis

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] - \frac{1}{\sigma}\vec{\nabla}_p^2 Q - \frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\hat{k} \bullet \vec{\nabla}_p \times \vec{Fr}\right)$$

A B C D E

Now we need to **tackle** term C, which involves **much nastier algebra** using **vector identities** ($\vec{\nabla}^2\left(\vec{a} \bullet \vec{b}\right)$) and the **thermal wind** **equations**.

Doing out the **dot products** and **interchanging** the **orders of differentiation**, we **arrive at**:

$$-\frac{1}{\sigma}\vec{\nabla}_p^2\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(T\right)\right] = \frac{f_o}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p\left(\zeta_g\right) - \vec{V}_g \bullet \vec{\nabla}_p\frac{\partial}{\partial p}\left(\zeta_g\right) + \left(F_2\frac{\partial F_1}{\partial p} - F_1\frac{\partial F_2}{\partial p}\right)\right]$$

5 6 7

Terms 5 and 6 are the **same as** terms 1 and 3 (we'll come **back to them in a minute!**), and term 7 **contains** the **deformation terms**:

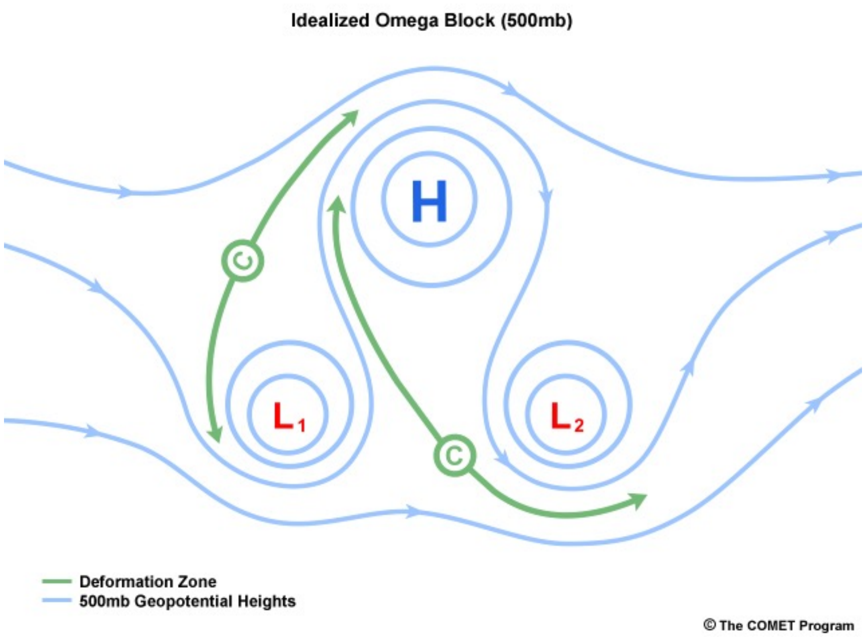
$$F_1 = \frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} \quad \text{and} \quad F_2 = \frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}$$

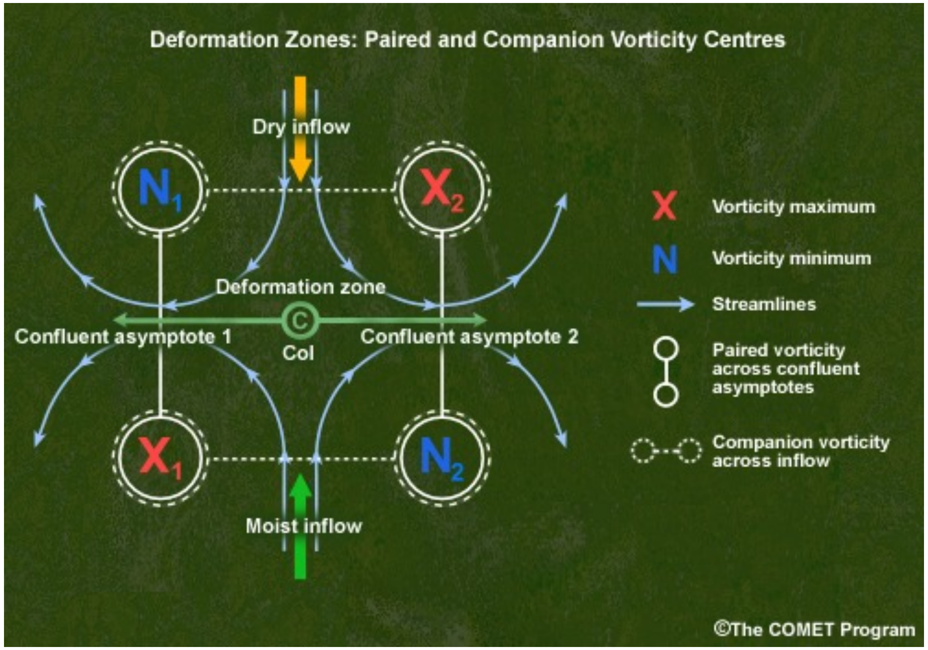
These **look similar to** **vorticity** and **divergence**, but **with opposite + and – signs**, and **have to do with** the **rate of change** of the **shape** of **fluid parcels**...**basically**, **distortion**!

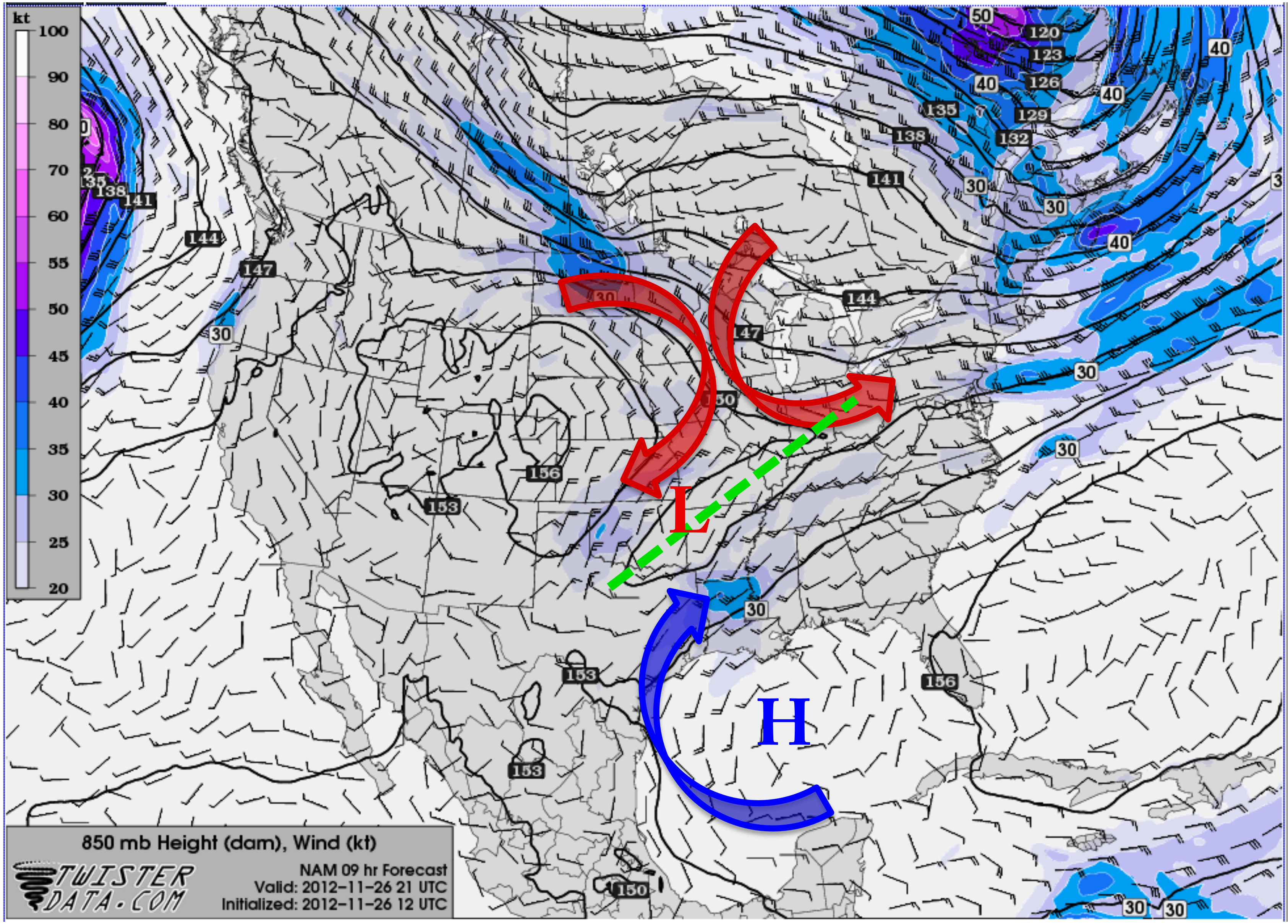
$$F_1 = \frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} \quad \text{and} \quad F_2 = \frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}$$

Lance Bosart: "Deformation is the red-headed stepchild of synoptic meteorology" = It's largely ignored!

This is because it is generally weak, except in the vicinity of fronts and jets.







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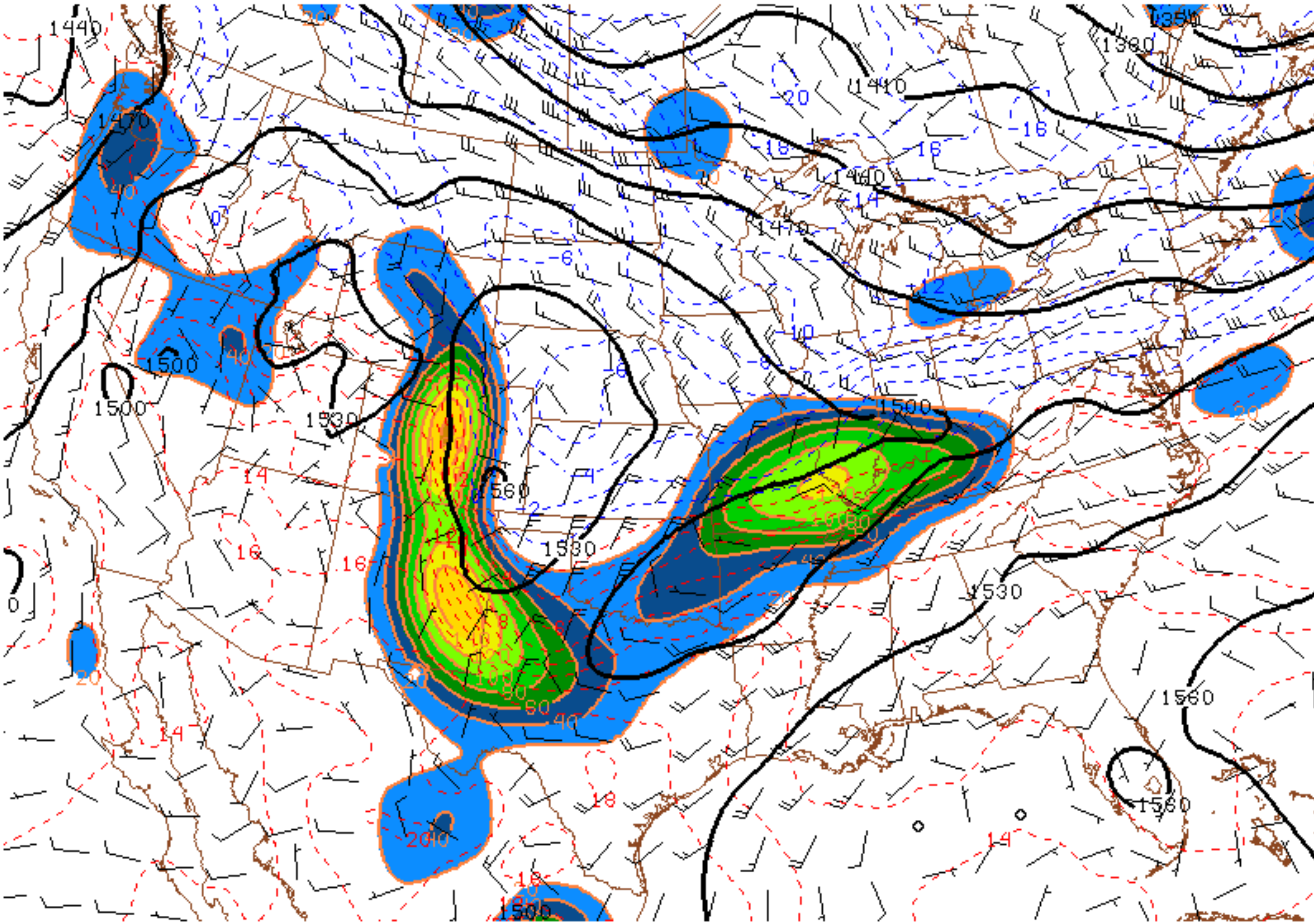
TWISTER DATA.COM
NAM 09 hr Forecast
Valid: 2012-11-26 21 UTC
Initialized: 2012-11-26 12 UTC

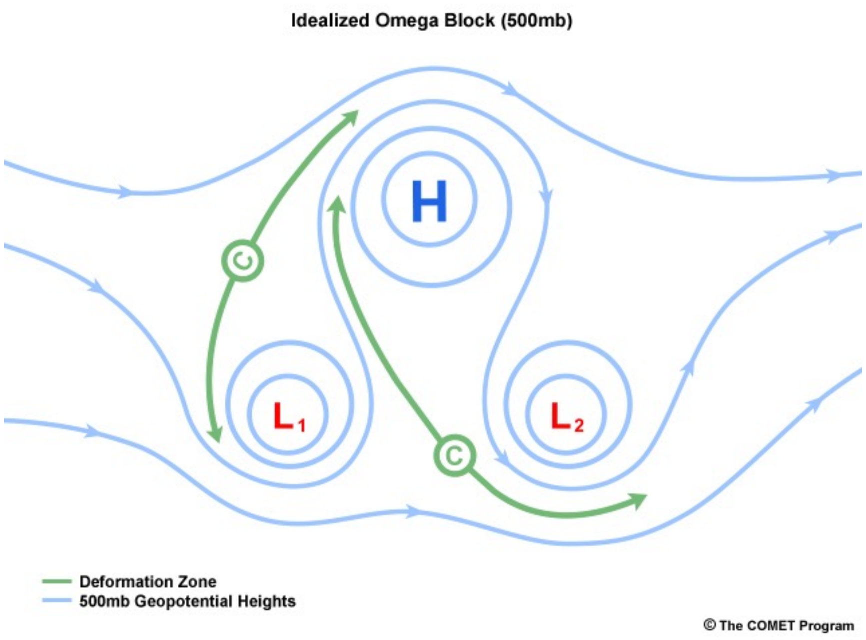


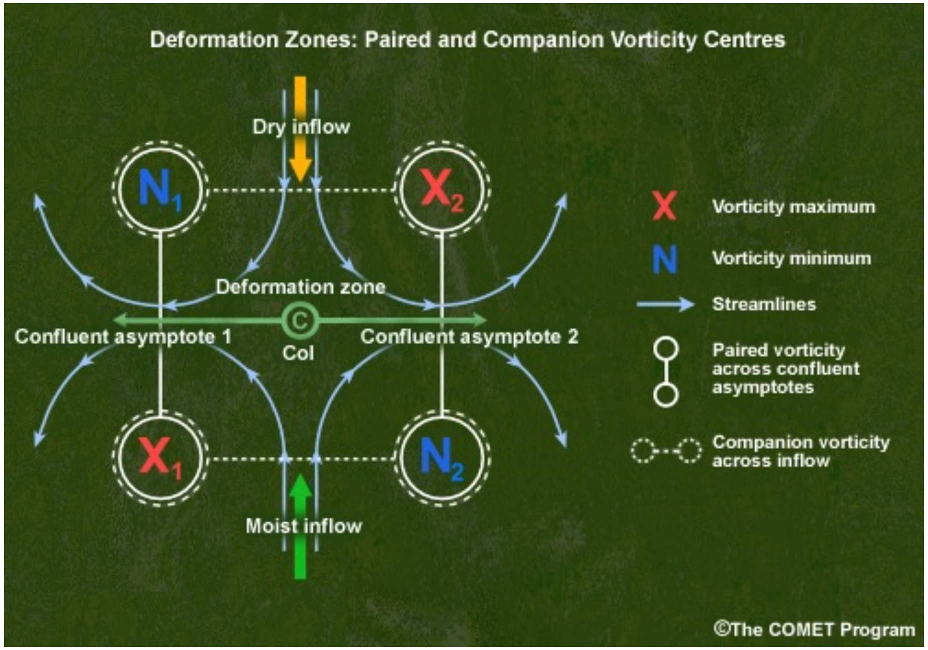
121126/2100V001 850 mb Petterssen frontogenesis (fill)
121126/2100V001 850 mb height, temperature and wind

$$F_1 = \frac{\partial u_g}{\partial x} - \frac{\partial v_g}{\partial y} \quad \text{and} \quad F_2 = \frac{\partial v_g}{\partial x} + \frac{\partial u_g}{\partial y}$$

Lance Bosart: "Deformation is the red-headed stepchild of synoptic meteorology" = It's largely ignored!

This is because it is generally weak, except in the vicinity of fronts and jets.





For our purposes, we will ignore deformation (term 7) and gather the remaining terms…

B + C = $\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p\left(\zeta_g \right)+\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p f+\frac{f_o}{\sigma}\vec{V}_g\bullet\vec{\nabla}_p\frac{\partial \zeta_g}{\partial p}+\frac{f_o}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p\left(\zeta_g \right)\right]-\frac{f_o}{\sigma}\vec{V}_g\bullet\vec{\nabla}_p\frac{\partial}{\partial p}\left(\zeta_g \right)$

1 2 3 5 6

We note terms 3 and 6 cancel, and terms 1 and 5 add to leave:

$$2\frac{f_o}{\sigma}\left[\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p\left(\zeta_g \right)\right]+\frac{f_o}{\sigma}\frac{\partial \vec{V}_g}{\partial p}\bullet\vec{\nabla}_p f$$

Usually, term 2 is small for midlatitude weather systems with a generally westerly (*i* component) thermal wind and north–south (*j* component) gradient of f.

Thus, we are left with ONE forcing term for vertical motion:

$$\left(\vec{\nabla}_p^2+\frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T\bullet\vec{\nabla}_p\zeta_g\right)$$

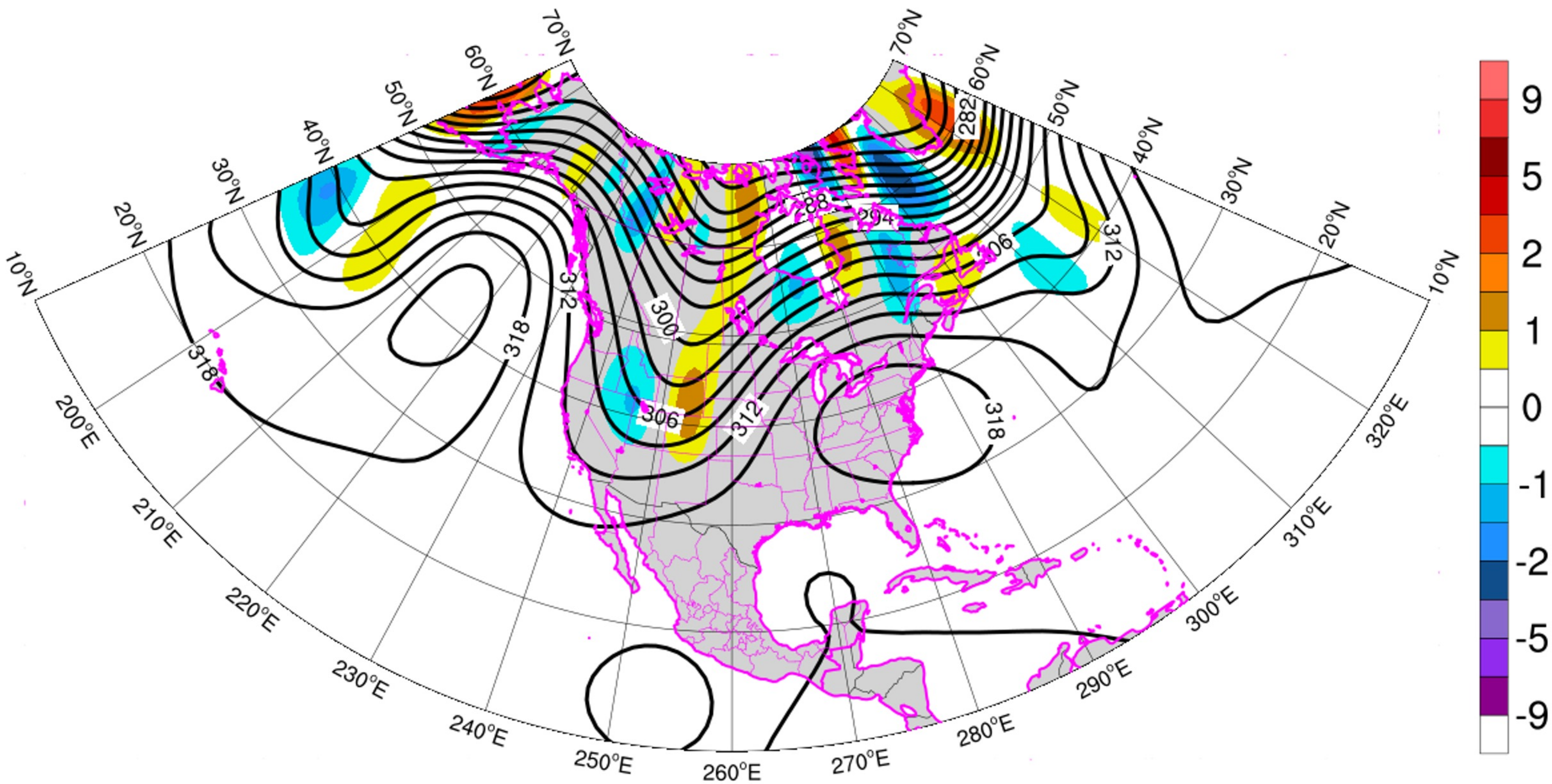
A B + C

The advection of geostrophic relative vorticity by the thermal wind!

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!



700 hPa Z and Differential Vorticity Advection Term (A) QG Omega Forcing (traditional form) (smoothed)
84-h GFS forecast at 2025100418 (2025100106 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

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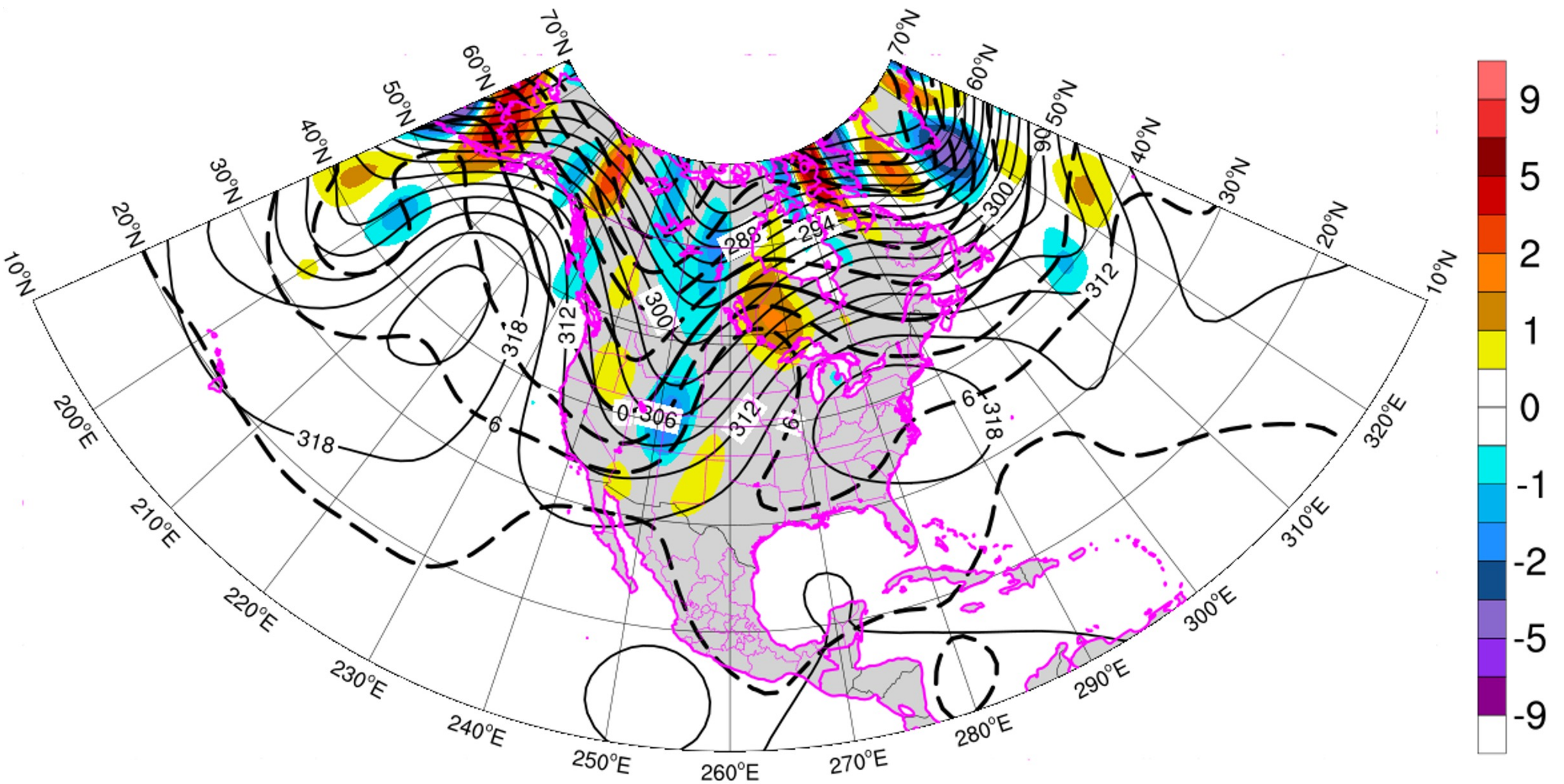

700 hPa Z, Temperature, and Thermal Advection Term (B) QG Omega Forcing (traditional form) (smoothed)
84-h GFS forecast at 2025100418 (2025100106 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

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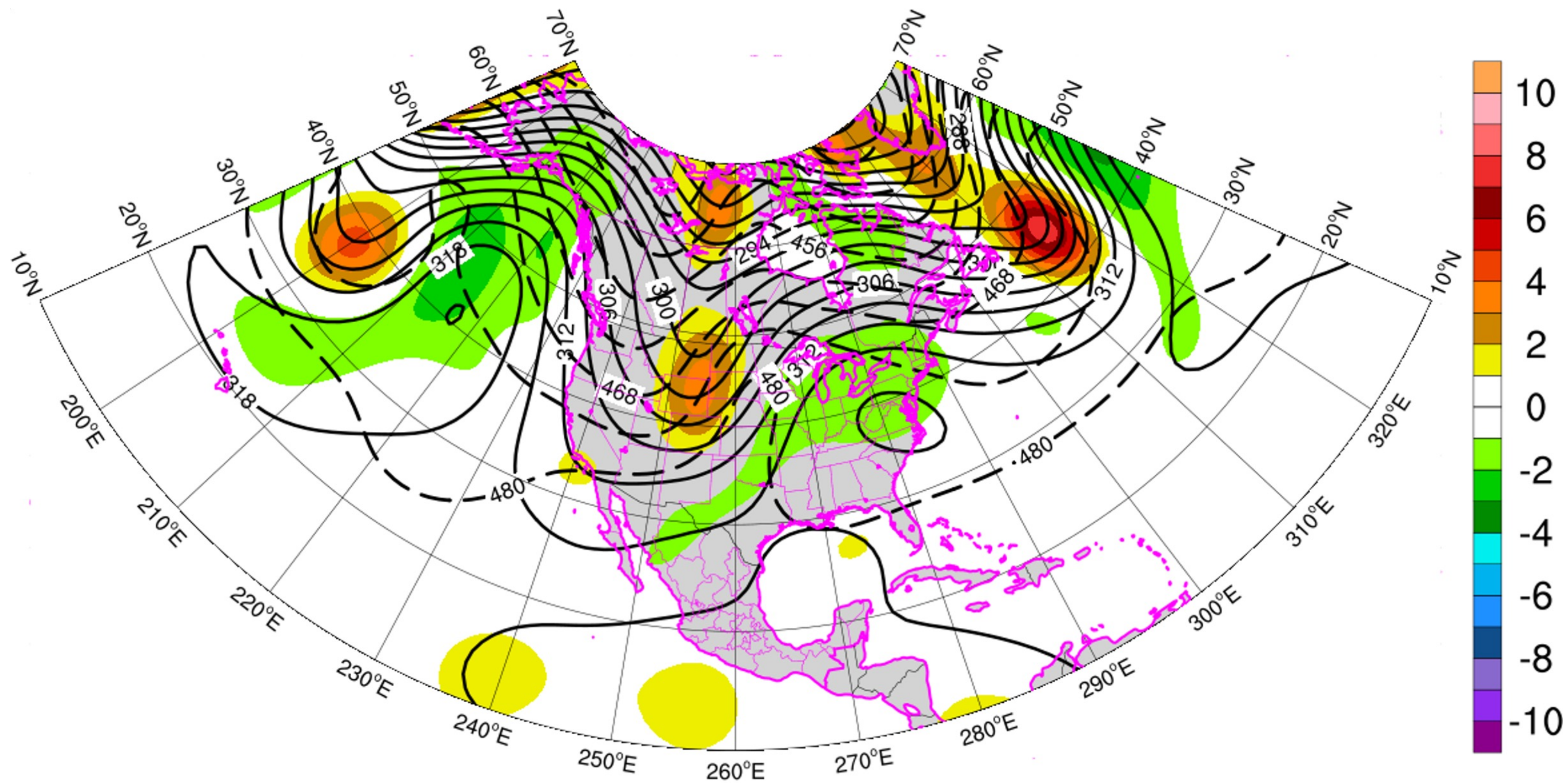
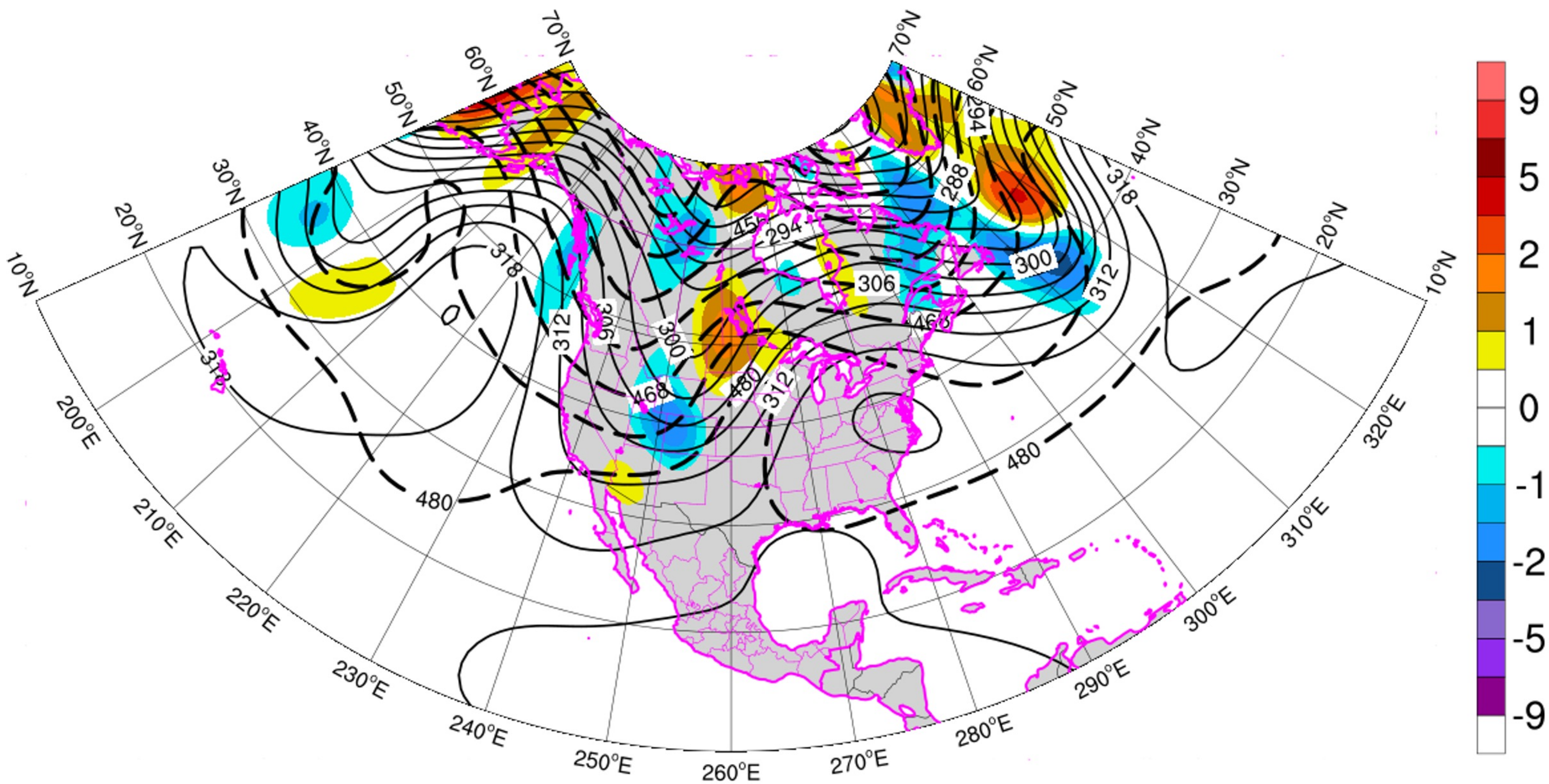


700 hPa Height, Geostrophic Vorticity, and 900-500 hPa Thickness (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!

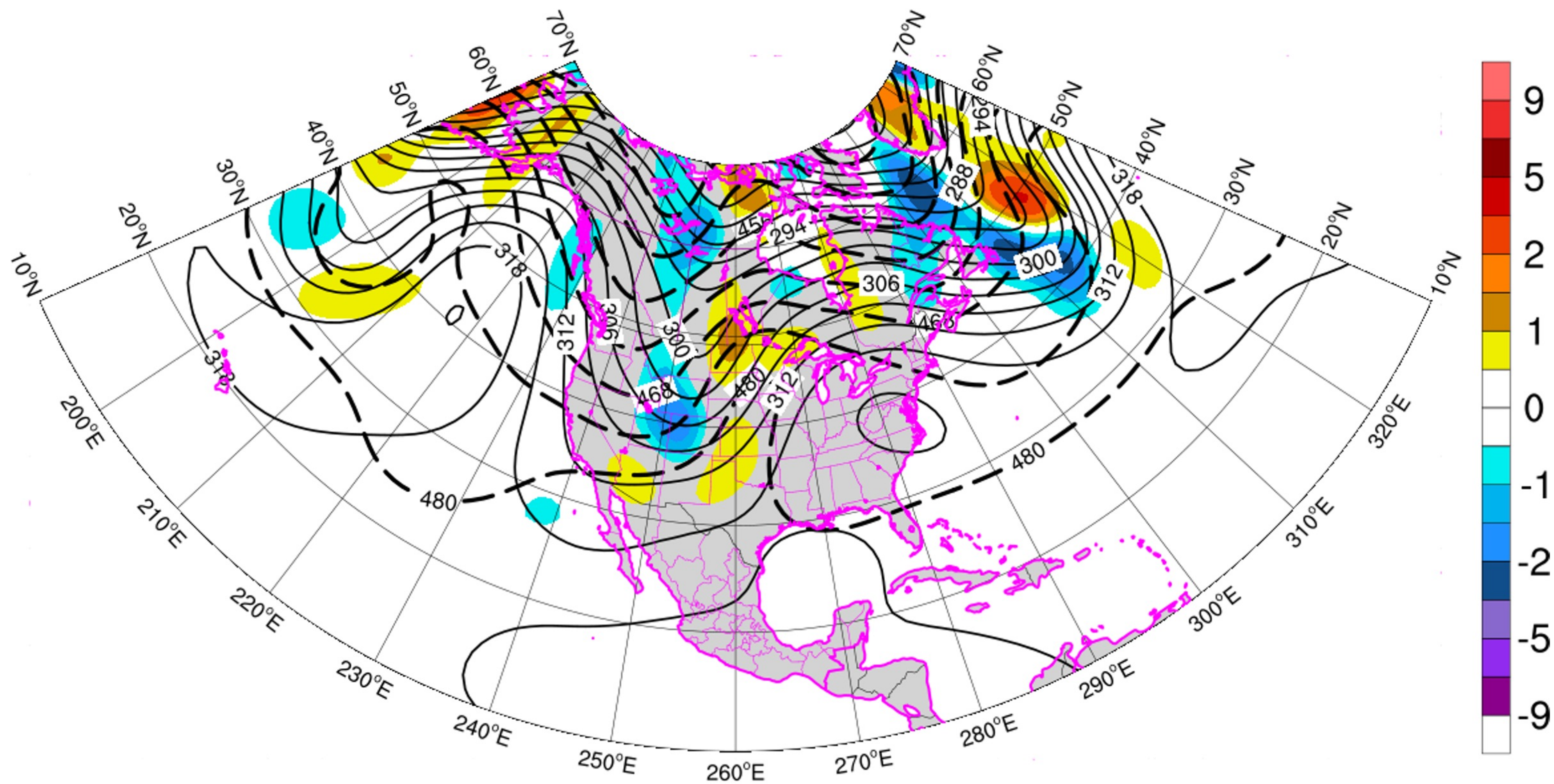


700 hPa Z, 900-500 hPa Thick, and Vort Advect Term (A) QG Omega Forcing (Trenberth form) (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!

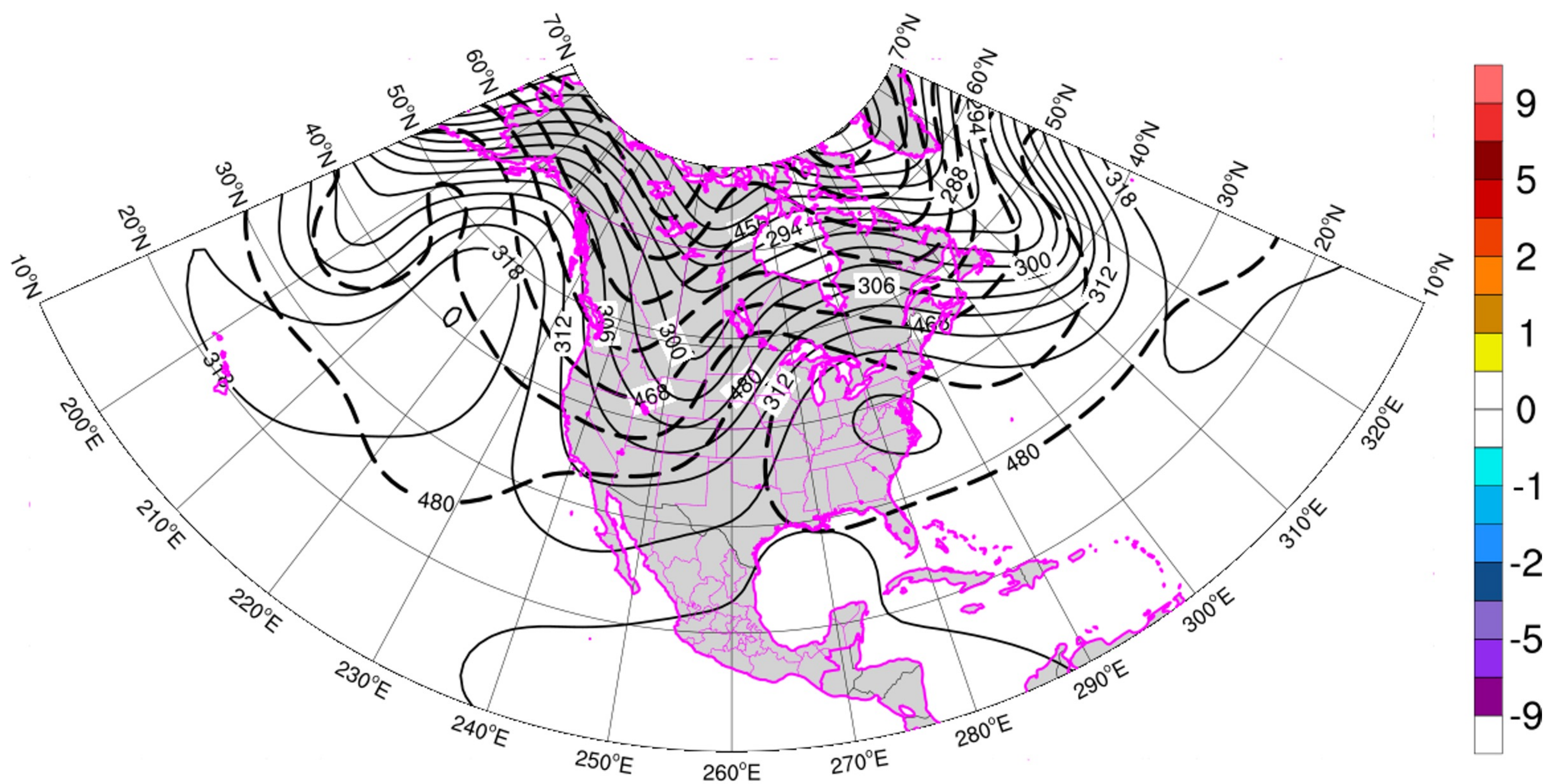


700 hPa Z, 900-500 hPa Thick, and Total RHS QG Omega Forcing (Trenberth form) (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!

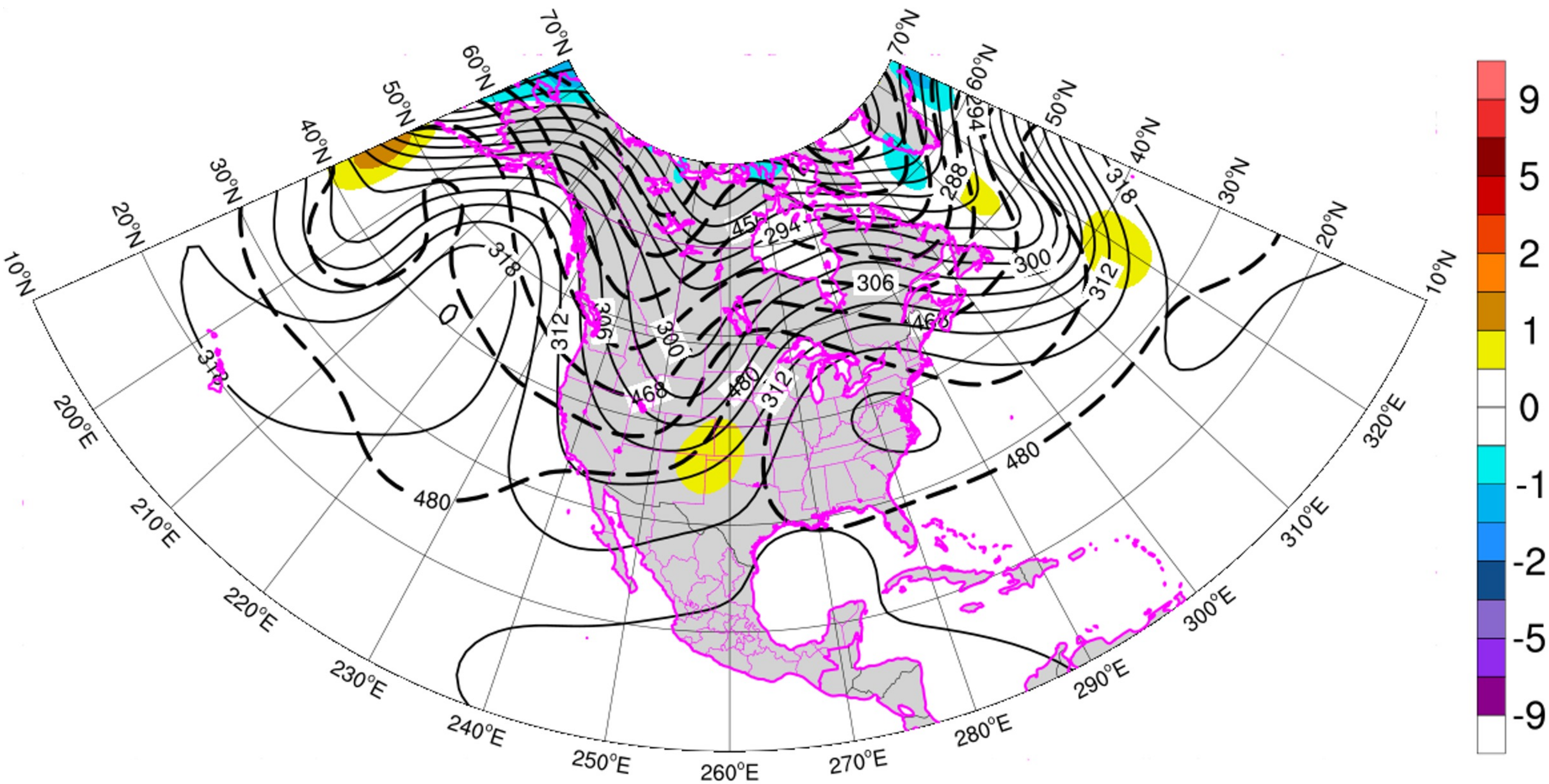


700 hPa Z, 900-500 hPa Thick, and f advect term B QG Omega Forcing (Trenberth form) (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!

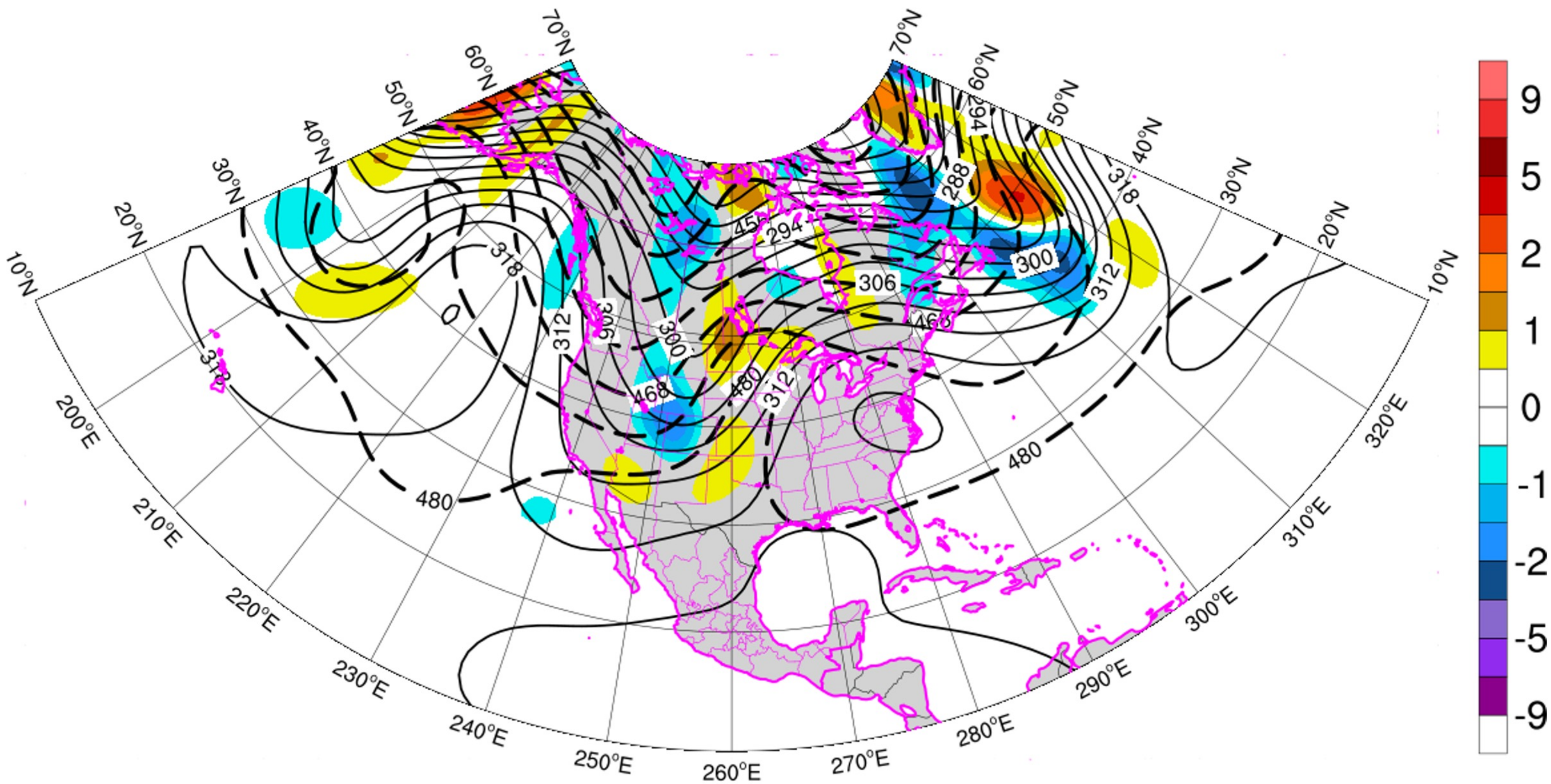


700 hPa Z, 900-500 hPa Thick, and deformation term C QG Omega Forcing (Trenberth form) (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)

$$\left(\vec{\nabla}_p^2 + \frac{f_o^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = 2\frac{f_o}{\sigma}\left(-\vec{V}_T \bullet \vec{\nabla}_p \zeta_g\right)$$

To evaluate the forcing for vertical motion, we plot 1000–500-hPa thickness (for V_T) with 700-hPa heights/vorticity.

From ONE map, we can determine the QG forcing for ascent: CVA by the thermal wind will lead to upward motion!



700 hPa Z, 900-500 hPa Thick, and Total RHS QG Omega Forcing (Trenberth form) (smoothed)
102-h GFS forecast at 2025100418 (2025093012 init)