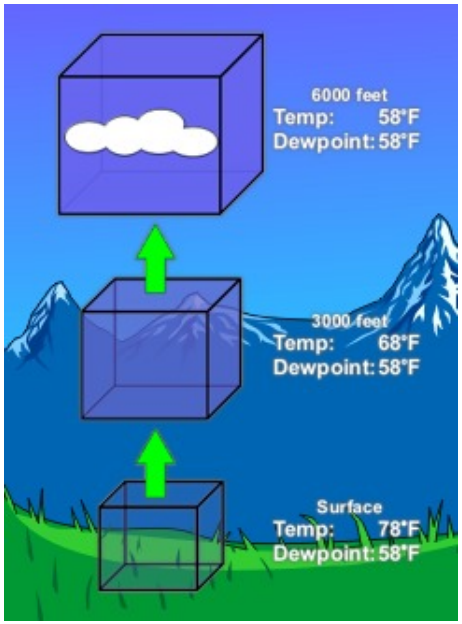
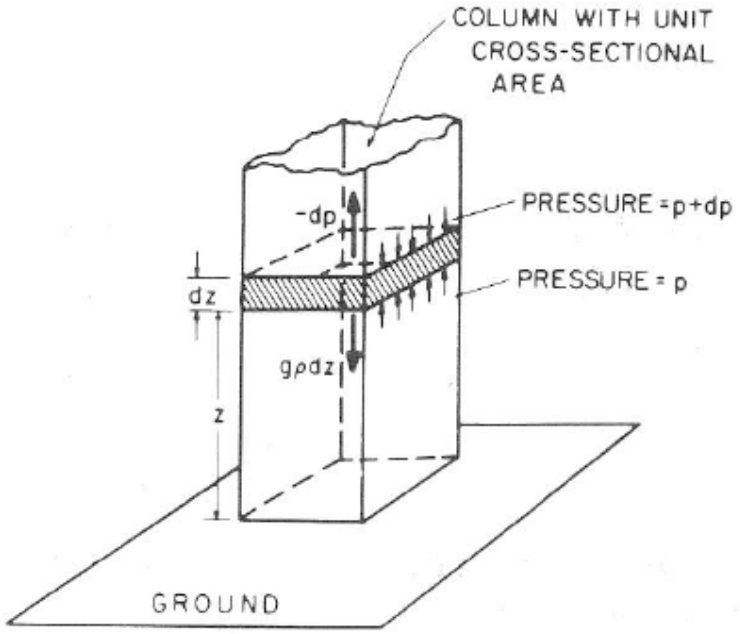
Frames of Reference and the Total Derivative

- The equations that express the basic conservation laws of atmospheric dynamics (conservation of mass, momentum, and energy) are usually derived considering an infinitesimal volume of air like you may have done when deriving the hydrostatic relationship or thinking about adiabatic processes.





- We may consider this volume in one of two ways:

  1) The three-dimensional volume (usually a cube) can be considered fixed in space. This frame of reference is called the Eulerian frame and the rate of change of mass or energy in the volume is due to the motion of air through the boundaries of the cube.

  2) The volume can freely move with the motion of the air, always consisting of the same mass of "tagged" air molecules. This is called the Lagrangian frame of reference.

~~~ Consider the OCEAN example on the next page! ~~~
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Eulerian Current Observations

Measurements of ocean current are collected using a variety of methods. One popular way to measure ocean currents is to determine the water's velocity at one fixed place in the ocean. This type of measurement is called **Eulerian**, in honor of the Swiss mathematician Leonhard Euler. This is typically accomplished using an electro-mechanical current meter (which measures the velocity at a single depth) or Acoustic Doppler Current Profiler (ADCP) (which can provide a profile of velocity with depth). Current meters are usually on a wire of a mooring, which is deployed from a ship. ADCPs can be mounted on a mooring, the ocean bottom, or the underside of a vessel. Both will provide a time series of the velocity of the ocean's water at a single geographic location. Current measurements are also obtained using High-Frequency Radar.

Lagrangian Current Observations

Drifting profiler shown before deployment. Instruments of this type make Lagrangian measurements as they are swept along by ocean currents. Satellites track their positions with Global Positioning Systems (GPS).

Another direct way to measure ocean currents is by tagging a water material with either floats or dyes. This viewpoint of following a tagged water parcel is called **Lagrangian**, named in honor of Joseph Louis Lagrange, a French mathematician. Near-surface ocean currents are measured by so-called drifters, which is a buoy that rides at the ocean surface and is usually weighted at some depth to negate the direct effects of wind on the buoy itself. Tracking this drifter (by satellite, radar, radio, sound, etc.) will give a description of the ocean current.

The animation below shows the tracks of many drifters through time. Each half second frame of the animation represents 12 hours of drifter movement. The animation loops after representing two weeks of drifter movement. Notice the drifter dragged along by the Loop Current until it is ejected through the Straits of Florida and picked up by the Gulf Stream. Also notice the drifter trapped in a mid-Atlantic Gyre at the upper right of the map.

The short and straight arrows near shore represent current measurements taken by fixed Eulerian stations either on shore, on ocean towers, or on moored bouys. The changing splotches of color off the northern coast of North Carolina and the southeastern tip of Florida represent hundreds of simultaneous current measurements taken by high frequency radar. The lively background colors represent sea surface temperature as measured by satellites. More red means warmer water. More blue means cooler water.

Notice how the drifter pulled along by the Loop Current is following the edge of a temperature gradient in the Gulf of Mexico. The Loop Current trapping a mass of warmer water. Notice how the Gulf Stream pushes warmer water from the tropics northward along the coast. The Gulf Stream is part of the Global Conveyor Belt distributing heat around the world by means of flowing ocean currents.

Scientists often use Eulerian and Lagrangian measurements together to learn about the flowing ocean. Scientists have launched thousands of Lagrangian drifters into the ocean and deployed hundreds of fixed Eulerian stations near shore to observe ocean circulation. The animation below superimposing Eulerian and Lagrangian measurements was constructed by Jesse Cleary of the North Carolina Coastal Ocean Observing System, a SEACOOS partner.

- **Both frames have their advantages and significant place in dynamics, and thus we want to find a mathematical expression relating the two frames; i.e., an equation relating the rate of change of a variable (Θ for example) following the motion to the rate of change at a point.**
- **The rate of change following the motion (Lagrangian frame) is called the substantial, material, or total derivative and is noted D / Dt or d / dt.**
- **The rate of change at a fixed point (Eulerian frame) is called the local derivative and is noted $\partial / \partial t$ (the partial derivative).**
- **Consider an air parcel with $\Theta = \Theta(x_0, y_0, z_0, t_0)$ and note how Θ changes as the parcel moves to a new point $x = x_0 + dx$, $y = y_0 + dy$, and $z = z_0 + dz$ at $t = t_0 + dt$.**

- The change in Θ following the motion between the two points is

$$d\Theta = \left(\frac{\partial \Theta}{\partial t}\right) dt + \left(\frac{\partial \Theta}{\partial x}\right) dx + \left(\frac{\partial \Theta}{\partial y}\right) dy + \left(\frac{\partial \Theta}{\partial z}\right) dz$$

- Dividing through by dt we have

$$\frac{d\Theta}{dt} = \frac{\partial \Theta}{\partial t} + \left(\frac{\partial \Theta}{\partial x}\right)\frac{dx}{dt} + \left(\frac{\partial \Theta}{\partial y}\right)\frac{dy}{dt} + \left(\frac{\partial \Theta}{\partial z}\right)\frac{dz}{dt}$$

This is the change in Θ following the motion (Lagrangian).

This is the local (Eulerian) time rate of change.

These terms are the movement of the parcel over a span of time, i.e., the u, v, and w components of the wind.

- **With these realizations, we may rewrite the equation as**

$$\frac{d\Theta}{dt} = \frac{\partial\Theta}{\partial t} + \left(\frac{\partial\Theta}{\partial x}\right)u + \left(\frac{\partial\Theta}{\partial y}\right)v + \left(\frac{\partial\Theta}{\partial z}\right)w$$

- **We also recognize that** $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$, $\vec{\nabla}\Theta = \frac{\partial\Theta}{\partial x}\hat{i} + \frac{\partial\Theta}{\partial y}\hat{j} + \frac{\partial\Theta}{\partial z}\hat{k}$,

  and recall $\vec{a} \bullet \vec{b} = a_i b_i + a_j b_j + a_k b_k$ **, so that we may write:**

$$\frac{d\Theta}{dt} = \frac{\partial\Theta}{\partial t} + \vec{V} \bullet \vec{\nabla}\Theta$$

- **This equation says that the total change of Θ following the motion is equal to the local rate of change of Θ (like heating from the sun rising) plus the** $\vec{V} \bullet \vec{\nabla}\Theta$ **term which is advection, the change in Θ owing to the motion of the air.**

- We usually rewrite the equation to have the local change (what we actually measure at fixed locations like the thermometer at KALB) on the left hand side, such that

$$\frac{\partial \Theta}{\partial t} = \frac{D\Theta}{Dt} - \vec{V} \bullet \vec{\nabla}\Theta$$

- Now taking a closer look at the advection term, recall from our vector review that

$$\vec{a} \bullet \vec{b} = a_i b_i + a_j b_j + a_k b_k = \left|\vec{a}\right|\left|\vec{b}\right|\cos\alpha$$

- With this in mind, consider the following figures:

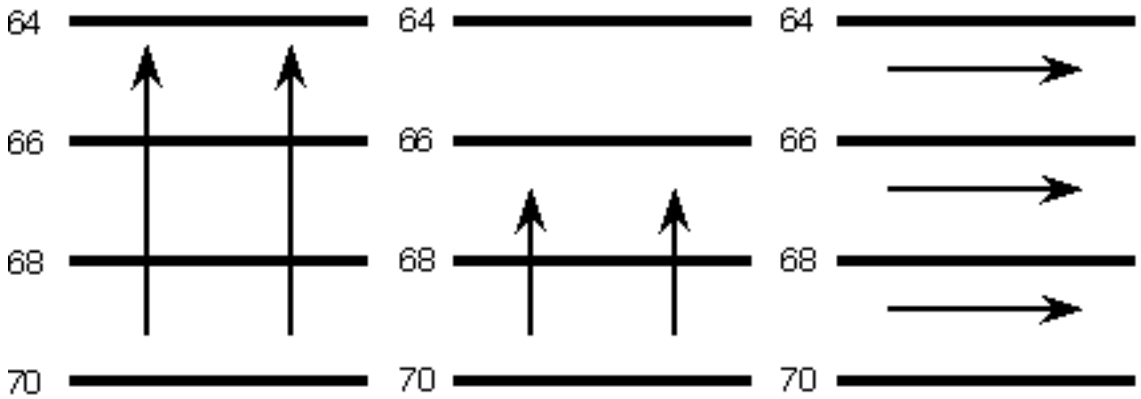
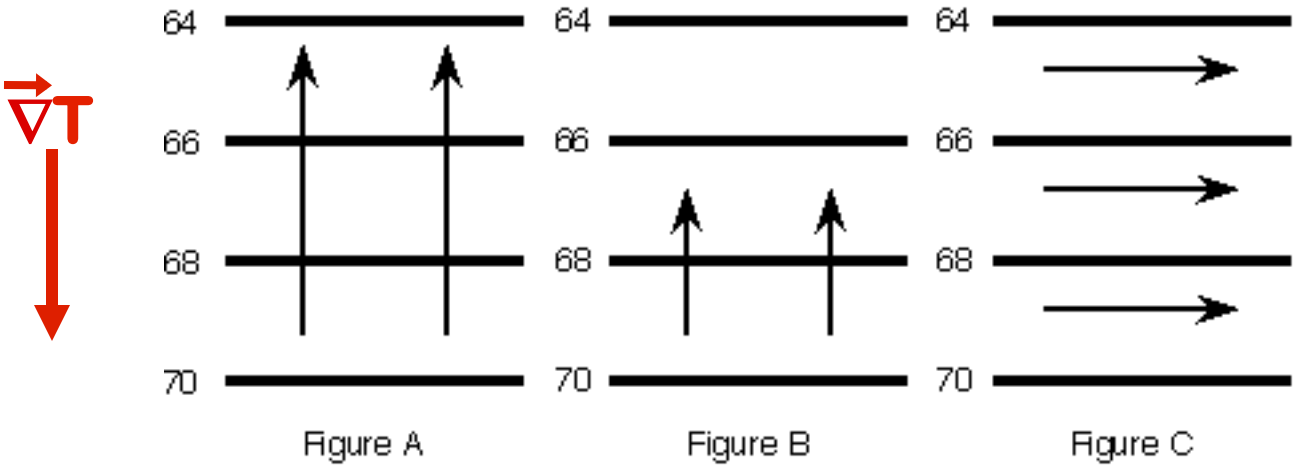


Figure A Figure B Figure C



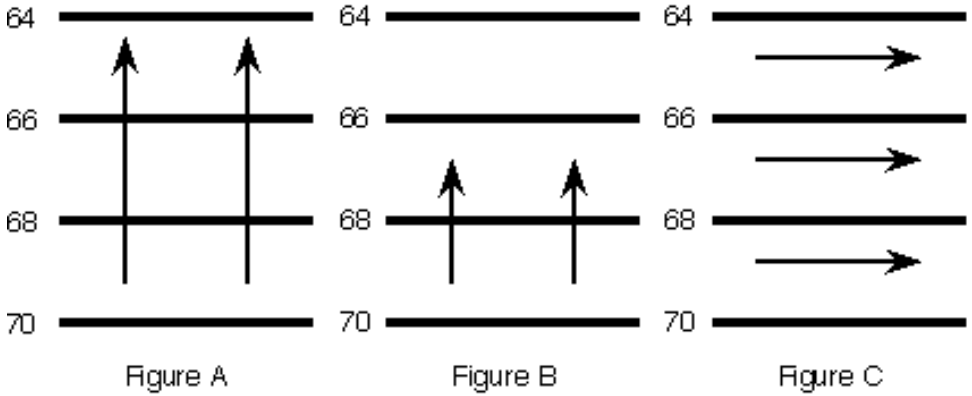
- In all panels, the $\vec{\nabla}T$ points down (to the south), towards warmer temperatures.
- Thus, in Figures A and B, the southerly winds ($v > 0$; perpendicular to the isotherms) are blowing warm air northward. This is called warm air advection.
- In Figure C, the winds are blowing parallel to the isotherms and thus there is no advection.
- Does this make sense in terms of our equation?

$$-\vec{V} \bullet \vec{\nabla}T = -\left|\vec{V}\right|\left|\vec{\nabla}T\right|\cos\alpha$$

- **Yes! In A and B, the angle between the winds and the $\vec{\nabla}T$ is 180°. Thus**

$$-\vec{V} \bullet \vec{\nabla}T = -\left|\vec{V}\right|\left|\vec{\nabla}T\right|\cos 180^o > 0$$

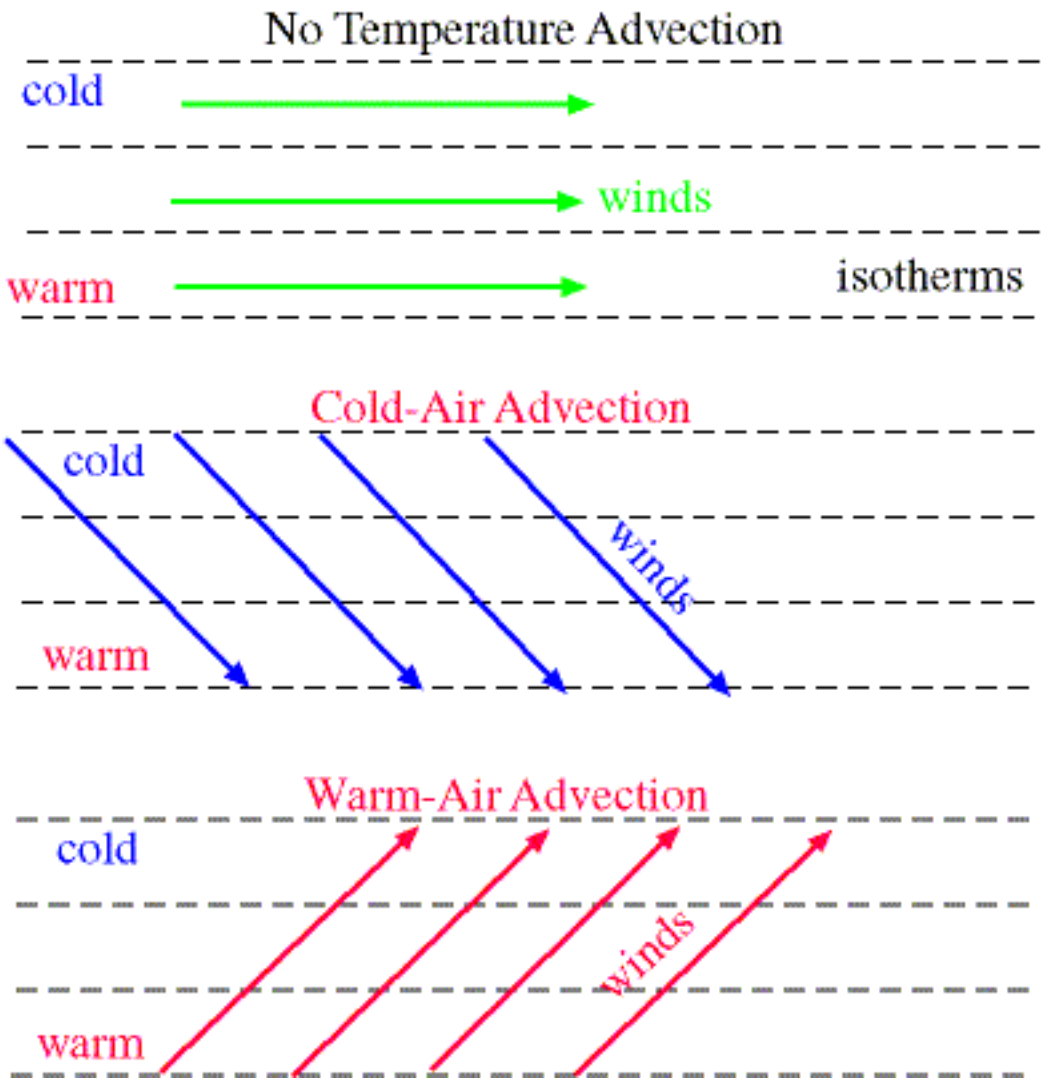
and the warm air advection contributes positively to the local temperature change!



- **In Figure C, the angle between the winds and the $\vec{\nabla}T$ is 90°. Thus**

$$-\vec{V} \bullet \vec{\nabla}T = -\left|\vec{V}\right|\left|\vec{\nabla}T\right|\cos 90^o = 0$$

and there is no local temperature change due to advection.



$$-\vec{V} \bullet \vec{\nabla} T = -\left|\vec{V}\right|\left|\vec{\nabla} T\right| \cos 90^{o} = 0$$

$$-\vec{V} \bullet \vec{\nabla} T = -\left|\vec{V}\right|\left|\vec{\nabla} T\right| \cos 45^{o} < 0$$

$$-\vec{V} \bullet \vec{\nabla} T = -\left|\vec{V}\right|\left|\vec{\nabla} T\right| \cos 135^{o} > 0$$