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Quasi-Geostrophic Height Tendency Equation

1. Traditional form:

A diagnostic equation for geopotential height tendency $\left(\chi = \frac{\partial \Phi}{\partial t}\right)$ can be derived from combining the quasi-geostrophic vorticity and thermodynamic equations. The resulting expression, neglecting friction and diabatic effects, is known as the quasi-geostrophic height tendency equation [eq. 5.6.13 in Bluestein (1992), p. 330]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi = \underbrace{f_0\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right]}_{A} \underbrace{- \frac{f_0^2}{\sigma}\frac{\partial}{\partial p}\left[\frac{R}{p}\left(-\vec{V}_g \bullet \vec{\nabla}_p T\right)\right]}_{B}.$$

Note that the above form of the quasi-geostrophic geopotential height tendency equation from Bluestein (1992) is equivalent to the versions found in eq. 6.23 in Holton (2004; p. 157) and eq. 8.8 in Martin (2006; p. 250). Holton (2004) and Martin (2006) express geostrophic relative vorticity and temperature in terms of geopotential. Also for term B above, Holton (2004) places $\frac{f_0^2}{\sigma}$ inside the $\frac{\partial}{\partial p}$, but since σ is assumed constant in the calculation here, the Bluestein (1992) and Holton (2004) forms are equivalent. The graphics on this web page would be qualitatively identical for all three forms of the equation.

Term A: Advection of geostrophic absolute vorticity by the geostrophic wind. Cyclonic (anticyclonic) vorticity advection is associated with geopotential height falls (rises). This term **acts as the propagation mechanism for troughs and ridges**.
Term B: Differential advection of temperature by the geostrophic wind. Warm air advection increasing (decreasing) with height is associated with height falls (rises), while cold air advection increasing (decreasing) with height is associated with height rises (falls). This term **acts as the amplification mechanism for troughs and ridges**.

Synoptic Application: Regions of 500 hPa height falls (rises) are associated with a positive (negative) contribution from the right-hand-side. For the NCEP–GFS forecasts shown on this page, term A is evaluated at 500 hPa and term B is evaluated using finite differencing between 700 and 300 hPa. This methodology indicates the *forcing* for geopotential height falls/rises at 500 hPa, not *actual* height falls/rises since we do not evaluate the 3-D Laplacian on the left-hand-side.

Further Reading: Bluestein (1992), pp. 329–330; Holton (2004), pp. 157–159; Martin (2006), pp. 246–250.

2. Quasi-Geostrophic Potential Vorticity (QGPV) form:

An alternate form of the diagnostic equation for geopotential height tendency, which arises from combining and manipulating the frictionless form of the quasi-geostrophic vorticity equation and adiabatic form of the quasi-geostrophic thermodynamic equation, can be written using QGPV [eq. 5.8.15 in Bluestein (1992), p. 373]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\chi = f_0\left(-\vec{V}_g \bullet \vec{\nabla}_p q\right)$$

where q is defined as [eq. 5.8.10 in Bluestein (1992), p. 372]

$$q = QGPV = \underbrace{\zeta_g + f}_{\eta} + \underbrace{\frac{\partial}{\partial p}\left(\frac{f_0}{\sigma}\frac{\partial \Phi}{\partial p}\right)}_{stability}.$$

The geostrophic absolute vorticity term (η) was calculated at 500 hPa and the stability term was calculated as described in Hakim et al. (1995; pp. 2665–2666). The QGPV was multiplied by $-g\left(\frac{d\theta_{REF}}{dp}\right)$ to convert QGPV in units of s^{-1} to standard potential vorticity units (PVU) as described in Hakim et al. (1995) and Bosart et al. (1996). The lapse rate for a standard atmosphere was computed from the equation for a pressure dependent static stability from Keyser et al. (1988; p. 767) $\sigma = -h\frac{d\theta_{REF}}{dp}$. Static stability is assumed constant ($\sigma=2.0\times10^{-6}$ m^2 s^{-2} Pa^{-2}) for our purposes, and $h = \left(\frac{R}{p_0}\right)\left(\frac{p_0}{p}\right)^{\frac{C_v}{C_p}}$ where p_0 = 1000 hPa and p = 500 hPa resulting in $\frac{d\theta_{REF}}{dp}=-4.25\times10^{-4}$ K Pa^{-1}.

Note that the above form of the QGPV equation from Bluestein (1992) is equivalent to the versions found in eq. 6.25 in Holton (2004; p. 160) and eq. 9.9d in Martin (2006; p. 284). Holton (2004) and Martin (2006) express geostrophic relative vorticity in terms of geopotential. Also for the stability term, Martin (2006) places $\frac{f_0}{\sigma}$ outside the $\frac{\partial}{\partial p}$, but since σ is assumed constant in the calculation here, the Bluestein (1992) and Martin (2006) forms are equivalent. The graphics on this web page would be qualitatively identical for all three forms of the equation.

Synoptic Application: Since QGPV is conserved following geostrophic motion, regions of positive (negative) QGPV advection by the 500 hPa geostrophic wind are associated with 500 hPa height falls (rises). This methodology indicates the *forcing* for geopotential height falls/rises at 500 hPa, not *actual* height falls/rises since we do not evaluate the 3-D Laplacian on the left-hand-side. The QGPV form of the QG height tendency equation avoids the possibility of canceling effects between vorticity advection and differential thermal advection in the traditional form of the QG height tendency equation by containing only one forcing function on the right-hand-side.

Further Reading: Bluestein (1992), pp. 370–373; Holton (2004), pp. 159–161; Martin (2006), pp. 284–285.

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Key for symbols:

$\vec{\nabla}_p$	gradient on a pressure surface (m^{-1})
∇_p^2	Laplacian operator on a pressure surface (m^{-2})
$\vec{V}_g$	geostrophic wind ($m\ s^{-1}$)
ζ_g	geostrophic relative vorticity (s^{-1})
η	geostrophic absolute vorticity (s^{-1})
f	Coriolis parameter (s^{-1}; $f_0=1.0\times10^{-4}\ s^{-1}$)
R	gas constant for dry air (R=287 J $K^{-1}\ kg^{-1}$)
T	temperature (K)
σ	static stability (assumed constant $\sigma=2.0\times10^{-6}\ m^2\ Pa^{-2}\ s^{-2}$)

p pressure (Pa)
q quasi-geostrophic potential vorticity (PVU)
Φ geopotential ($m^2\ s^{-2}$)
g gravity at sea-level (9.81 $m\ s^{-2}$)
θ_{REF} potential temperature in standard atmosphere (K)