Thomas J. Galarneau, Jr.
Last updated 14 February 2010

Quasi-Geostrophic Omega Equation

1. Traditional form:

Much like the quasi-geostrophic height tendency equation, a diagnostic equation for vertical motion (ω) can be derived by manipulating the quasi-geostrophic vorticity and thermodynamic equations for adiabatic, frictionless flow to yield [eq. 5.6.11 in Bluestein (1992), p. 329]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{-\frac{f_0}{\sigma}\frac{\partial}{\partial p}\left[-\vec{V}_g \bullet \vec{\nabla}_p\left(\zeta_g + f\right)\right]}_{A} + \underbrace{\frac{R}{\sigma p}\left[-\nabla_p^2\left(-\vec{V}_g \bullet \vec{\nabla}_p T\right)\right]}_{B}.$$

Note that the above form of the quasi-geostrophic omega equation from Bluestein (1992) is equivalent to the versions found in eq. 6.34 in Holton (2004; p. 165) and eq. 6.26 in Martin (2006; p. 162). Holton (2004) expresses differential vorticity and thermal advection in terms of geopotential, and Martin (2006) expresses thermal advection in terms of geopotential. Also, Martin (2006) places σ on the left-hand-side, but since σ is assumed constant here it essentially acts as a scaling factor only. The graphics on this web page would be qualitatively identical for all three forms of the equation.

Term A: Differential advection of geostrophic absolute vorticity by the geostrophic wind. Cyclonic vorticity advection increasing (decreasing) with height is associated with upward (downward) vertical motion, while anticyclonic vorticity advection increasing (decreasing) with height is associated with downward (upward) vertical motion.

Term B: Horizontal Laplacian of temperature advection. Warm (cold) air advection is associated with upward (downward) vertical motion.

Synoptic Application: Regions of 700 hPa upward (downward) vertical motion are associated with a positive (negative) contribution from the right-hand-side. For the NCEP–GFS forecasts shown on this page, term A is evaluated using finite differencing between 1000 and 500 hPa and term B is evaluated at 700 hPa. This methodology indicates *forcing* for vertical motion at 700 hPa, not *actual* vertical motion since we do not evaluate the 3-D Laplacian on the left-hand-side.

Further Reading: Bluestein (1992), pp. 328–346; Holton (2004), pp. 164–165; Martin (2006), pp. 160–164.

2. Sutcliffe-Trenberth form:

Trenberth (1978) demonstrated that the forcing functions on the right-hand-side of the quasi-geostrophic omega equation can be manipulated and simplified, resulting in the following form [eq. 5.7.40 in Bluestein (1992), p. 349]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{f_0}{\sigma}\left[\underbrace{2\left(\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p \zeta_g\right)}_{A} + \underbrace{\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p f}_{B} \underbrace{- 2D^2 \frac{\partial \theta_D}{\partial p}}_{C}\right].$$

Term A: Advection of geostrophic relative vorticity by the thermal wind. Cyclonic (anticyclonic) geostrophic relative vorticity advection by the thermal wind is associated with upward (downward) vertical motion.
Term B: Advection of planetary vorticity by the thermal wind. Cyclonic (anticyclonic) planetary vorticity advection by the thermal wind is associated with upward (downward) motion.
Term C: Deformation and thermal deformation.

Terms B and C are usually small compared to term A in the middle and upper-troposphere since (1) the thermal wind vector is westerly on average while the Coriolis parameter varies meridionally only, and (2) geostrophic temperature advection is weak in the middle and upper troposphere (untrue near baroclinic zones). Therefore, the Trenberth formulation of the QG omega equation can be rewritten neglecting terms B and C resulting in the Sutcliffe-Trenberth form [eq. 5.7.42 in Bluestein (1992), p. 349]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega \approx \frac{f_0}{\sigma}2\left(\frac{\partial \vec{V}_g}{\partial p} \bullet \vec{\nabla}_p \zeta_g\right).$$

The Sutcliffe-Trenberth forms presented in eq. 6.36* in Holton (2004, p. 166) and in eq. 6.32 in Martin (2006, p. 165) retain the contribution of advection of planetary vorticity by the thermal wind (term B above). Also, Martin (2006) places σ on the left-hand-side, but since σ is assumed constant here it essentially acts as a scaling factor only. The graphics on this web page would be qualitatively similar for all three forms of the equation.

Synoptic Application: Regions of 700 hPa upward (downward) vertical motion are associated with a positive (negative) contribution from the right-hand-side. For the NCEP–GFS forecasts shown on this page, term A is evaluated using 700 hPa

* The coefficient "2" on the right-hand-side is omitted in eq. 6.36 in Holton (2004).

geostrophic relative vorticity and the 1000–500 hPa thermal wind. This methodology indicates the *forcing* for vertical motion at 700 hPa, not the *actual* vertical motion since we do not evaluate the 3-D Laplacian on the left-hand-side. The Sutcliffe-Trenberth form of the QG omega equation is preferred over the traditional form for forecasting because it (1) is Galilean invariant, (2) eliminates the cancellation problem between terms A and B in the traditional form, and (3) is easy to estimate by simply plotting 700 hPa geopotential heights overlaid by 1000–500 hPa thickness contours.

Further Reading: Bluestein (1992), pp. 346–350; Holton (2004), pp. 165–168; Martin (2006), pp. 164–166; Trenberth (1978).

3. Q-Vector form:

Hoskins et al. (1978) demonstrated that the right-hand-side of the traditional form of the quasi-geostrophic omega equation can be rewritten as [eq. 5.7.56 in Bluestein (1992), p. 353]

$$\left(\nabla_p^2 + \frac{f_0^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \underbrace{-2\vec{\nabla}_p \bullet \vec{Q}}_{A} \underbrace{- \frac{R}{\sigma p}\beta\frac{\partial T}{\partial x}}_{B}$$

where the **Q**-vector is defined as [eq. 5.7.55 in Bluestein (1992), p. 352]

$$\vec{Q} = -\frac{R}{\sigma p}\begin{pmatrix} \dfrac{\partial \vec{V}_g}{\partial x} \bullet \vec{\nabla}_p T \\ \dfrac{\partial \vec{V}_g}{\partial y} \bullet \vec{\nabla}_p T \end{pmatrix} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}.$$

Holton (2004, p. 170) writes the **Q**-vector form of the omega equation (eq. 6.53) with σ on the left-hand-side, and term B in terms of wind shear. The **Q**-vector equation (eq. 6.54) is written as above less the σ parameter, which was moved to the left-hand-side of the omega equation. Martin (2006, p. 170) writes the **Q**-vector form of the omega equation (eq. 6.40) less the Beta term and with σ on the left-hand-side. The **Q**-vector equation (eq. 6.41) is written in terms of geopotential and less the σ parameter, which was moved to the left-hand-side of the omega equation. Since σ is assumed constant here it essentially acts as a scaling factor only. The graphics on this web page would be qualitatively similar for all three forms of the equation.

Term A: Divergence of the **Q**-vector. Regions of 700 hPa upward (downward) vertical motion are associated with **Q**-vector convergence (divergence).
Term B: Beta term. Usually neglected because it is small relative to term A.

Synoptic Application: For tropospheric-deep synoptic systems, the **Q**-vector points in the direction of the lower-tropospheric horizontal branch of the ageostrophic secondary circulation needed to restore thermal wind balance. In the upper troposphere, the **Q**-vector and the horizontal branch of the ageostrophic secondary circulation point in opposite directions. Here, the **Q**-vector is evaluated at 700 hPa and points toward regions of upward vertical motion at 700 hPa. This methodology indicates the *forcing* for vertical motion at 700 hPa, not the *actual* vertical motion since we do not evaluate the 3-D Laplacian on the left-hand-side.

The **Q**-vector form of the QG omega equation has the following advantages over the traditional form when forecasting [as written in Bluestein (1992), p. 353]:

1. The forcing function can be evaluated on a single pressure surface, while the traditional form requires information at two pressure surfaces.
2. Galilean invariant.
3. Eliminates the cancellation problem in the traditional form.
4. No term has been neglected.
5. **Q**-vectors can be plotted on geopotential height and temperature analyses/forecasts to obtain a representation of vertical motion and the ageostrophic wind.

Further Reading: Bluestein (1992), pp. 350–368; Holton (2004), pp. 168–172; Martin (2006), pp. 166–181; Hoskins et al. (1978); Sanders and Hoskins (1990); Durran and Snellman (1987).

References:
Bluestein, H. B., 1992: *Principles of Kinematics and Dynamics*. Vol. I. *Synoptic-Dynamic Meteorology in Midlatitudes*. Oxford University Press, 431 pp.
Durran, D. R., and L. W. Snellman, 1987: The diagnosis of synoptic-scale vertical motion in an operational environment. *Wea. Forecasting,* **2,** 17–31.
Holton, J. R., 2004: *An Introduction to Dynamic Meteorology.* 4th ed. Academic Press, 535 pp.
Hoskins, B. J., I. Draghici, and H. C. Davies, 1978: A new look at the omega equation. *Quart. J. Roy. Meteor. Soc.,* **104,** 31–38.
Sanders, F., and B. J. Hoskins, 1990: An easy method for estimation of Q-vectors on weather maps. *Wea. Forecasting,* **5,** 346–353.
Martin, J. E., 2006: *Mid-Latitude Atmospheric Dynamics: A First Course.* John Wiley & Sons, Ltd, 324 pp.
Trenberth, K. E., 1978: On the interpretation of the diagnostic quasi-geostrophic omega equation. *Mon. Wea. Rev.,* **106,** 131–137.

Key for symbols:

$\vec{\nabla}_p$	gradient on a pressure surface (m^{-1})
∇_p^2	Laplacian operator on a pressure surface (m^{-2})
$\vec{V}_g$	geostrophic wind ($m\ s^{-1}$)
ζ_g	geostrophic relative vorticity (s^{-1})
f	Coriolis parameter (s^{-1}; f_0=1.0×10^{-4} s^{-1})
R	gas constant for dry air (R=287 J K^{-1} kg^{-1})
T	temperature (K)
σ	static stability (assumed constant σ =2.0×10^{-6} m^2 Pa^{-2} s^{-2})
p	pressure (Pa)
β	Beta=df/dy (m^{-1} s^{-1})