Lecture 6: Quasigeostrophic Rossby waves

In Lecture 3, we derived the properties of Rossby waves in the reduced-gravity model, linearised about a state of rest. Our key findings were:

- Rossby waves rely on the β-effect for their existence;
- their zonal phase propagation is westward;
- their zonal group propagation is westward for long waves and eastward for short waves.

We also discussed the mechanism for Rossby wave propagation through potential vorticity conservation.

Since Rossby waves occur at low Rossby number, it is reasonable to try and model them using the quasigeostrophic equations.

Carl-Gustaf Rossby (1898-1957)

6.1 Potential vorticity equation

Consider small-amplitude disturbances to a uniform zonal background flow $(U, 0, 0)$ for which the streamfunction is

$$\Psi = -Uy$$

and the quasigeostrophic potential vorticity is

$$Q = \beta y.$$

The streamfunction for the total flow (background plus small disturbance) is thus

$$\psi = -Uy + \psi',$$

and the potential vorticity

$$q = \beta y + q'$$

where

$$q' = \nabla_h^2 \psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right).$$

We now substitute the above in the quasigeostrophic potential vorticity equation (5.15) and neglect terms that are quadratic in the disturbance quantities:

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right) q' + v'\frac{\partial Q}{\partial y} = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\left\{\nabla_h^2 \psi' + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi'}{\partial z}\right)\right\} + \beta\frac{\partial \psi'}{\partial x} = 0. \qquad (6.1)$$

The first term represents the rate of change of the perturbed potential vorticity following the mean flow, and the second term represents the advection of mean potential vorticity by the perturbed flow.

6.2 Dispersion relation

We now look for plane-wave solutions,

$$\psi' = Ae^{i(kx+ly+mz-\omega t)}.$$

We shall also assume constant stratification, i.e., $N^2 = \text{constant}$.

This gives the dispersion relation for quasigeostrophic Rossby waves:

$$\omega = Uk - \frac{\beta k}{k^2 + l^2 + \frac{f_0^2 m^2}{N^2}}. \tag{6.2}$$

As in Lecture 3, the β-effect is crucial for these Rossby waves. Setting $\beta = 0$ gives $\omega = Uk$, which implies $c_p^{(x)} = c_g^{(x)} = U$, i.e., the waves are merely carried along by the background flow, U.

More generally, the zonal phase speed of the waves is

$$c_p^{(x)} = U - \frac{\beta}{k^2 + l^2 + \frac{f_0^2 m^2}{N^2}},$$

and hence

$$c_p^{(x)} - U < 0;$$

the wave crests and troughs move *westward* relative to the background flow, although they can move eastward relative to the ground for a sufficiently strong background flow. The latter explains why mid-latitude atmospheric weather systems generally propagate eastward.

6.3 Vertical propagation

For given real values of k and l (e.g., imposed by land-sea contrasts or orography), we obtain vertical propagation when m is real and nonzero.

Vertical propagation therefore corresponds to $m^2 > 0$, i.e.,

$$U - c_p^{(x)} = \frac{\beta}{k^2 + l^2 + \frac{f_0^2 m^2}{N^2}} < U_c$$

where the critical background velocity,

$$U_c = \frac{\beta}{k^2 + l^2},$$

depends on the horizontal wavelengths of the wave.

Thus for vertical propagation we must have

$$0 < U - c_p^{(x)} < U_c. \tag{6.3}$$

In particular, for *stationary waves*, whose crests and troughs do not move relative to the ground such that $c_p^{(x)} = 0$, we obtain the *Charney-Drazin criterion*,

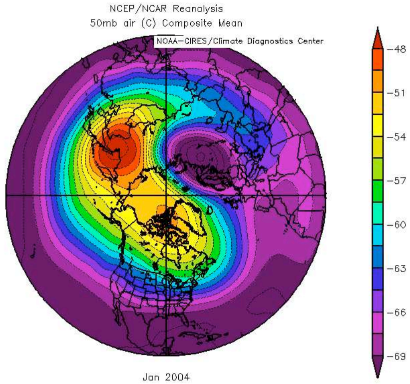
$$0 < U < U_c. \tag{6.4}$$

Stationary waves propagate vertically only in eastward background flows ($U > 0$) that are not too strong ($U < U_c$).

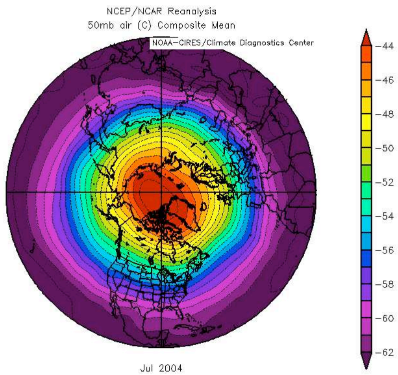
Moreover, since U_c increases with increasing horizontal wavelength, *long waves* propagate vertically under a wider range of eastward flows than *short waves*.

This is consistent with observations: *(θ on $p = 50\,\mathrm{mb}$, McLandress 2005)*

- In the winter stratosphere, winds are eastward and stationary Rossby waves have large horizontal scales.



- In the summer stratosphere, winds are westward and stationary Rossby waves are absent.



Rossby waves are strongly dispersive: an initial disturbance composed of a number of wavelengths will tend to break up, or disperse, as the different wavelength components propagate away at different phase speeds.

We can define the group velocity:

$$\mathbf{c}_g = (c_g^{(x)}, c_g^{(y)}, c_g^{(z)}) = \left(\frac{\partial \omega}{\partial k}, \frac{\partial \omega}{\partial l}, \frac{\partial \omega}{\partial m}\right).$$

For stationary waves, we put $c_p = \omega = 0$ *after* differentiation — note that stationary waves can still propagate information, even though their phase surfaces do not move; this is another consequence of their dispersive nature.

In particular,

$$c_g^{(z)} = \frac{\partial \omega}{\partial m} = \frac{2 f_0^2 \beta k m}{N^2 \left(k^2 + l^2 + \frac{f_0^2 m^2}{N^2}\right)^2}.$$

Since k, $m > 0$, we find $c_g^{(z)} > 0$, i.e., the waves propagate information upwards.

For upward propagating Rossby waves, the phase surfaces,

$$kx + ly + mz - \omega t = \text{constant},$$

slope westward with height; this is also observed for stratospheric Rossby waves.

6.4 Vertically-trapped Rossby waves

Returning to the dispersion relation (6.2), if $k^2 + l^2 > \beta/(U - c_p^{(x)})$ then m is imaginary.

If we set $m = i\mu$, where μ is real and positive, then the wave solutions are of the form

$$\psi' = Ae^{i(kx+ly-\omega t)}e^{-\mu z},$$

i.e., the waves propagate horizontally but not vertically, and decay with height.

Equivalently, this happens if

$$U - c_p^{(x)} < 0 \quad \text{or} \quad U - c_p^{(x)} > U_c.$$

For stationary waves ($c_p^{(x)} = 0$), vertical trapping occurs if the background winds are *westward* ($U < 0$) or *strongly eastward* ($U > U_c$).

6.5 Vertical modes

In the ocean, Rossby waves are strongly constrained by the surface and bottom boundaries. As in Lecture 4 for internal gravity waves, we can seek Rossby wave solutions with modal structures in the vertical.

Thus we seek solutions of the form

$$\psi'(x, y, z, t) = A(z)e^{i(kx+ly-\omega t)}.$$

Substituting the above into the potential vorticity equation (6.1) gives

$$\left(\frac{\beta k}{Uk - \omega} - k^2 - l^2\right) A + \frac{f_0^2}{N^2}\frac{d^2 A}{dz^2} = 0.$$

This separable solution only works if both

$$\left(\frac{\beta k}{Uk - \omega} - k^2 - l^2\right) = \lambda^2 \quad \text{and} \quad \frac{d^2 A}{dz^2} = -\lambda^2 \frac{N^2}{f_0^2} A$$

where λ is a separation constant.

Here we shall consider only waves in which the surface and bottom density anomalies vanish, giving the boundary conditions:

$$\frac{\partial \psi'}{\partial z} = 0 \quad \Rightarrow \frac{dA}{dz} = 0 \quad (z = 0,\ -H)$$

where $z = 0$ is the sea surface and $z = -H$ is the sea floor.

As in Lecture 4, the vertical modes satisfy

$$A = A_0 \cos\left(\frac{\lambda N}{f_0} z\right),$$

$$\text{where} \quad \frac{\lambda N H}{f_0} = n\pi \quad n = 0,\ 1,\ 2,\ \ldots$$

The corresponding dispersion relation for the waves is

$$\omega = Uk - \frac{\beta k}{k^2 + l^2 + \frac{n^2\pi^2 f_0^2}{N^2 H^2}}. \qquad (6.5)$$

This is equivalent to the dispersion relation for Rossby waves in the reduced-gravity model (3.6) if we set

$$L_d^{(n)} = \frac{NH}{n\pi f_0}, \qquad (6.6)$$

known as the *Rossby deformation radius* for the nth mode.

The deformation radius is infinite for the *barotropic mode* ($n = 0$), and increasingly smaller for each *baroclinic mode* ($n \geq 1$).

The first baroclinic deformation radius is approximately 1000 km in the atmosphere and 30 km in the ocean.

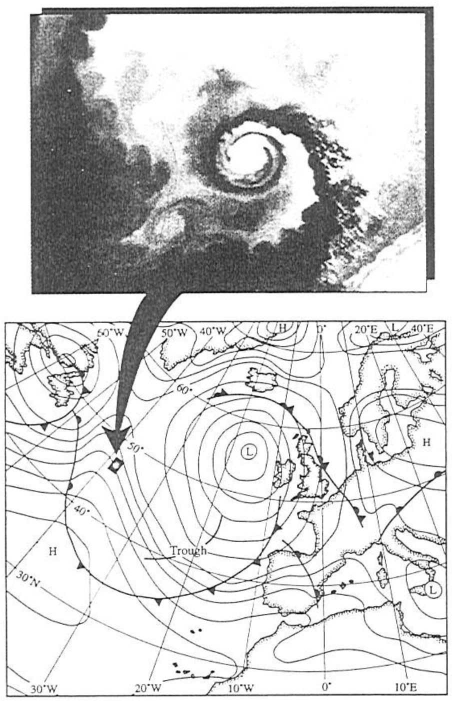


Illustration of the different deformation radii in the atmosphere and ocean as revealed through (nonlinear, breaking) Rossby waves.