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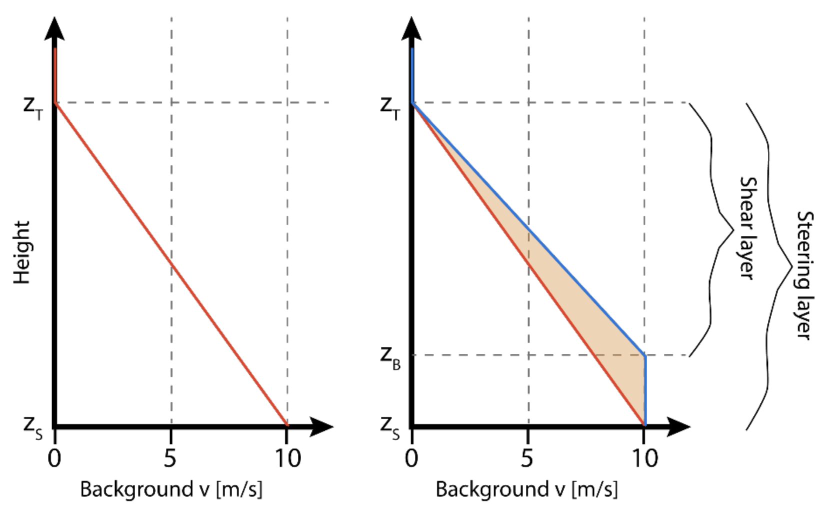
03/2022, transcribed 06/2023

CM1 can be used to spin-up and test TC responses to user-specified profiles of background flow. The background flow can be adjusted over time, as well. We can leverage these capabilities to test how TCBL structure is influenced by the joint forcings of storm motion and environmental vertical shear. There are countless ways to specify the background wind profile, but it is probably most insightful to start with simple specifications.

First, let us consider a few constraining elements for our problem:

1. The CM1 user is unable to specify TC motion simply through, for instance, u and v wind components. However, **we can apply a proxy for TC motion: column-averaged background flow.** The primary driving mechanism behind storm motion is likely to be the aggregate effect of steering flow through the depth of the vortex, and we can assume the steering flow is roughly the same as a non-zero background flow. A secondary element to storm motion is TC self-advection as the result of vortex tilt—for simplicity, we will assume this is negligible at the outset. Beta drift could also contribute to storm motion, but an f-plane will be applied to negate this effect.
2. Because the goal here is to examine BL structure in response to simple expressions of steering and shear, **we would prefer not to add background shear directly to the BL** (i.e., background winds kept uniform in the BL). We can restrict applied background shear to a layer that excludes the BL. However, as will be shown below, enforcing this rule requires an offsetting element to preserve the column-averaged background flow (i.e., the steering flow). This offset would also be necessary if the steering layer differs from the layer in which shear is applied.
3. Lastly, we prefer to have **background wind profiles that are continuous in the vertical**.

For example, suppose the user initially wants to run a case where the shear through the depth of the vortex between is 10 m/s, the background steering flow is 5 m/s, and the shear points antiparallel to the steering flow. Note that because the simulation will be on an f-plane, the model-relative direction is not relevant. Referring to constraint 1, the background wind profile of this initial approach could be represented by a piecewise linear function shown in red below. The bottom of the steering layer is the surface z_S, and the top of the steering layer is at the top of the vortex z_T. The shear vector through the depth of the vortex points in the negative y direction with a magnitude of 10 m/s, and the average background flow points in the positive y direction with a magnitude of 5 m/s, as intended. However, the user realizes that they would not like shear to be directly added to the BL by the background flow (constraint 2), so they define a shear layer with the bottom above the BL top at z_B and the top at z_T—the new wind profile is drawn in blue. The shear, now defined as the vector difference between z_T and z_B, is still appropriate, but now the average background flow between z_S and z_T is greater than 5 m/s.



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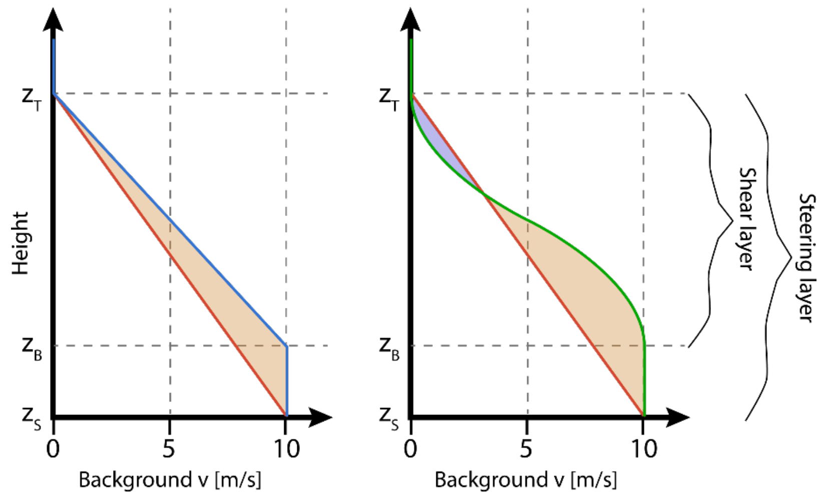
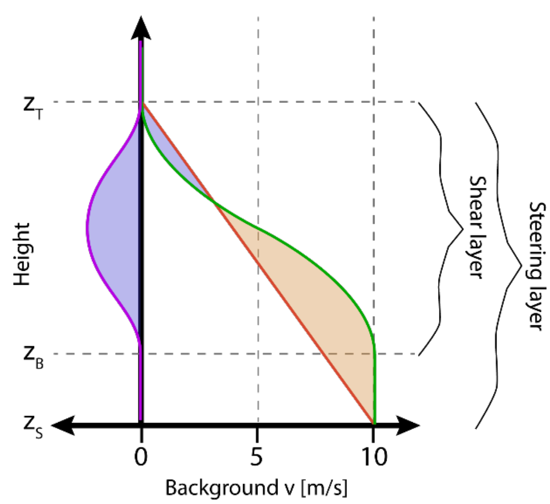
Aside from addressing the excess steering flow by imposing constraint 2, the wind profile used above has discontinuities at z_B and z_T that need to be resolved for constraint 3. One way to satisfy constraint 3 is to use a different piecewise function that uses wave segments instead of linear segments in the shear layer, as shown below by the green line:



The orange shading represents excess steering flow relative to the red linear profile (that satisfies constraint 1), while the blue shading represents a steering deficit relative to the red linear profile. Constraint 1 is satisfied if the sum of orange area (positive) and blue area (negative) equals zero.

To summarize the profiles devised so far, the red profile satisfies constraint 1, the blue profile satisfies constraint 2, and the green profile satisfies constraints 2 and 3. A simple way to modify the green profile to also satisfy constraint 1 is to shift the entire profile by a constant to adjust the average flow in the steering layer. Alternatively, we can add a second wave segment to the green profile that preserves continuity (constraint 3), preserves shear values (constraint 2), and when integrated, has equal and opposite area to the sum of the orange and blue areas shown above (constraint 1). Below, the second wave segment is shown in purple, and its integral through the steering layer (blue, negative area) is such that when combined with other blue and orange areas, the result is zero. By combining the purple and green profiles, we obtain a profile that satisfies all three constraints.



Now that we have drawn out our approach schematically, we will formulate expressions that can be applied easily to simulations.

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The prescribed background wind is a function of height z and is gradually nudged into a CM1 simulation. The steering flow is the average of background wind between the steering layer's bottom z_S and top z_D:

$$U = \frac{1}{z_D - z_S} \int_{z_S}^{z_D} \bar{u}(z) dz$$

$$V = \frac{1}{z_D - z_S} \int_{z_S}^{z_D} \bar{v}(z) dz$$

The user specifies the x-direction steering flow U, y-direction steering flow V, as well as z_S and z_D. The variables $\bar{u}$ and $\bar{v}$ are x-direction and y-direction background wind components. The background shear $\vec{S}$ is defined here as the difference between background wind at the top of the shear layer z_T and the bottom of the shear layer z_B:

$$\vec{S} = S_u \hat{\imath},\ S_v \hat{\jmath}$$

$$S_u = \bar{u}(z_T) - \bar{u}(z_B)$$

$$S_v = \bar{v}(z_T) - \bar{v}(z_B)$$

$$0 \le z_S \le z_B < z_T \le z_D \le z_{top}$$

The x- and y-components of background shear are S_u and S_v, respectively. The model top is z_{top}. The user specifies S_u, S_v, as well as z_T and z_B. Note that here we require the steering layer to include the entire shear layer. Background wind components are continuous throughout and uniform outside of the shear layer:

$$\bar{u}(z) = \begin{cases} \bar{u}(z_B), & z < z_B \\ F(z) = \sum_{i=1} f_i(z), & z_B \le z < z_T \\ \bar{u}(z_T), & z \ge z_T \end{cases}$$

$$\bar{v}(z) = \begin{cases} \bar{v}(z_B), & z < z_B \\ G(z) = \sum_{j=1} g_j(z), & z_B \le z < z_T \\ \bar{v}(z_T), & z \ge z_T \end{cases}$$

$$F(z_B) = \bar{u}(z_B)\,;\ F'(z_B) = 0$$

$$F(z_T) = \bar{u}(z_T)\,;\ F'(z_T) = 0$$

$$G(z_B) = \bar{v}(z_B)\,;\ G'(z_B) = 0$$

$$G(z_T) = \bar{v}(z_T)\,;\ G'(z_T) = 0$$

F and G are the x- and y-components of the background flow in the shear layer, respectively, and can be comprised of many waveforms so long as the boundary conditions at z_B and z_T are met. Primes denote vertical derivatives. The structure of each background wind component in the shear layer includes a primary wave—for simplicity, the wave used here is sinusoidal with an amplitude equal to half of the component's shear. Further, the primary wave has wavelength equal to double the shear layer thickness, and the height of the bottom of the shear layer is used to shift the wave phase. The primary waves can thus be expressed as:

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$$f_1(z) = \frac{\bar{u}(z_T) + \bar{u}(z_B) + S_u \cos[m_1(z - z_B)]}{2}$$

$$g_1(z) = \frac{\bar{v}(z_T) + \bar{v}(z_B) + S_v \cos[m_1(z - z_B)]}{2}$$

With wavenumber m_1:

$$m_1 = \frac{\pi}{z_T - z_B}$$

If we are content to shift the entire background profile uniformly to satisfy constraint 1, then we do not need to add more waves, and we can integrate $\bar{u}$ and $\bar{v}$ to get boundary values at the bottom and top of the shear layer:

$$U = \frac{1}{z_D - z_S}\left[\int_{z_S}^{z_B} \bar{u}(z_B)dz + \int_{z_B}^{z_T} f_1(z)dz + \int_{z_T}^{z_D} \bar{u}(z_T)dz\right]$$

$$U = \frac{z_B - z_S}{z_D - z_S}\,\bar{u}(z_B) + \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2(z_D - z_S)} + \frac{z_D - z_T}{z_D - z_S}\,\bar{u}(z_T)$$

$$(z_D - z_S)U = (z_B - z_S)\,\bar{u}(z_B) + \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2} + (z_D - z_T)\,\bar{u}(z_T)$$

$$(z_D - z_S)U = (z_B - z_S)\,\bar{u}(z_B) + \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2} + (z_D - z_T)\left(S_u + \bar{u}(z_B)\right)$$

$$(z_D - z_S)U = (z_B - z_S)\bar{u}(z_B) + \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2} + (z_D - z_T)S_u + (z_D - z_T)\bar{u}(z_B)$$

$$(z_D - z_S)U - \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2} - (z_D - z_T)S_u = (z_B - z_S)\bar{u}(z_B) + (z_D - z_T)\bar{u}(z_B)$$

$$\frac{2(z_D - z_S)U}{2} - \frac{(z_T - z_B)\left(S_u + 2\bar{u}(z_B)\right)}{2} - \frac{2(z_D - z_T)S_u}{2} = (z_D - z_S + z_B - z_T)\bar{u}(z_B)$$

$$\frac{2(z_D - z_S)U - (z_T - z_B)\,S_u - 2(z_T - z_B)\bar{u}(z_B) - 2(z_D - z_T)S_u}{2(z_D - z_S + z_B - z_T)} = \bar{u}(z_B)$$

$$2(z_D - z_S)U - (z_T - z_B)\,S_u - 2(z_D - z_T)S_u = 2(z_D - z_S + z_B - z_T)\bar{u}(z_B) + 2(z_T - z_B)\bar{u}(z_B)$$

$$2(z_D - z_S)U - (z_T - z_B)\,S_u - 2(z_D - z_T)S_u = 2(z_D - z_S)\bar{u}(z_B)$$

$$U - \frac{S_u}{2}\left[\frac{z_T - z_B + 2z_D - 2z_T}{z_D - z_S}\right] = \bar{u}(z_B)$$

$$U - \frac{S_u}{2}\left[\frac{2z_D - z_T - z_B}{z_D - z_S}\right] = \bar{u}(z_B)$$

$$V - \frac{S_v}{2}\left[\frac{2z_D - z_T - z_B}{z_D - z_S}\right] = \bar{v}(z_B)$$

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Alternatively, if we want to specify the shear layer's boundary conditions or confine adjustments to the shear layer to satisfy constraint 1, then a secondary wave is necessary to adjust the mean steering flow. For example, in the conceptual description, we chose to specify:

$$\bar{u}(z_B) = 10 \;\; ; \;\; \bar{u}(z_T) = 0$$

Or, more generally:

$$\bar{u}(z_B) = C \;\; ; \;\; \bar{u}(z_T) = C + S_u$$

where C is a known constant. We choose a secondary wave where it and its derivative are zero at the shear layer boundaries, so that when we add it to the primary wave, the boundary conditions are still satisfied. A simple wave that fits these requirements is a sinusoid displaced by its amplitude:

$$f_2(z) = A - A\cos[m_2(z - z_B)]$$

$$m_2 = \frac{2\pi}{z_T - z_B}$$

We can integrate $\bar{u}$ to find A:

$$U = \frac{1}{z_D - z_S}\left[\int_{z_S}^{z_B} C dz + \int_{z_B}^{z_T} \big(f_1(z) + f_2(z)\big) dz + \int_{z_T}^{z_D} (C + S_u) dz\right]$$

$$U = \frac{z_B - z_S}{z_D - z_S}\, C + \frac{(z_T - z_B)\,(S_u + 2C)}{2(z_D - z_S)} + \frac{z_T - z_B}{z_D - z_S} A + \frac{z_D - z_T}{z_D - z_S}\,(C + S_u)$$

$$U(z_D - z_S) = (z_B - z_S)C + \frac{(z_T - z_B)\,(S_u + 2C)}{2} + (z_T - z_B)A + (z_D - z_T)(C + S_u)$$

$$U(z_D - z_S) - (z_D - z_T)(C + S_u) - (z_B - z_S)C - \frac{(z_T - z_B)\,(S_u + 2C)}{2} = (z_T - z_B)A$$

$$\frac{U(z_D - z_S)}{z_T - z_B} - \frac{(z_D - z_T)(C + S_u)}{z_T - z_B} - \frac{(z_B - z_S)C}{z_T - z_B} - \frac{(S_u + 2C)}{2} = A$$

$$\frac{(z_D - z_S)U - (z_D - z_T + z_B - z_S)C - (z_D - z_T)S_u}{z_T - z_B} - \frac{(S_u + 2C)}{2} = A$$

Let's define the thicknesses of the steering and shear layers (Δ_U and Δ_S, respectively):

$$\Delta_U = z_D - z_S$$

$$\Delta_S = z_T - z_B$$

Thus:

$$\frac{U\Delta_U - (\Delta_U - \Delta_S)C - (z_D - z_T)S_u}{\Delta_S} - \frac{(S_u + 2C)}{2} = A$$

If $z_D = z_T$, $z_S = 0$, and $\bar{u}(z_T) = U + S_u/2$:

$$\frac{Uz_T - Cz_B}{\Delta_S} - \frac{(S_u + 2C)}{2} = A$$

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$$\frac{Uz_T - \bar{u}(z_T)z_B + S_u z_B}{\Delta_S} - \frac{S_u + 2\bar{u}(z_T) - 2S_u}{2} = A$$

$$\frac{Uz_T - Uz_B - (S_u z_B/2) + S_u z_B}{\Delta_S} - U = A$$

$$\frac{U\Delta_S + (S_u z_B/2)}{\Delta_S} - U = A$$

$$\frac{S_u z_B}{2\Delta_S} = A$$

… (continued)