Balanced Flows

- In addition to the geostrophic wind we discussed on Friday, there are additional balance expressions for the relationships among the wind, pressure, and temperature fields that are useful for analyzing atmospheric motions.

- Some of these relationships are most conveniently written and understood using coordinate systems other than those we have previously discussed.

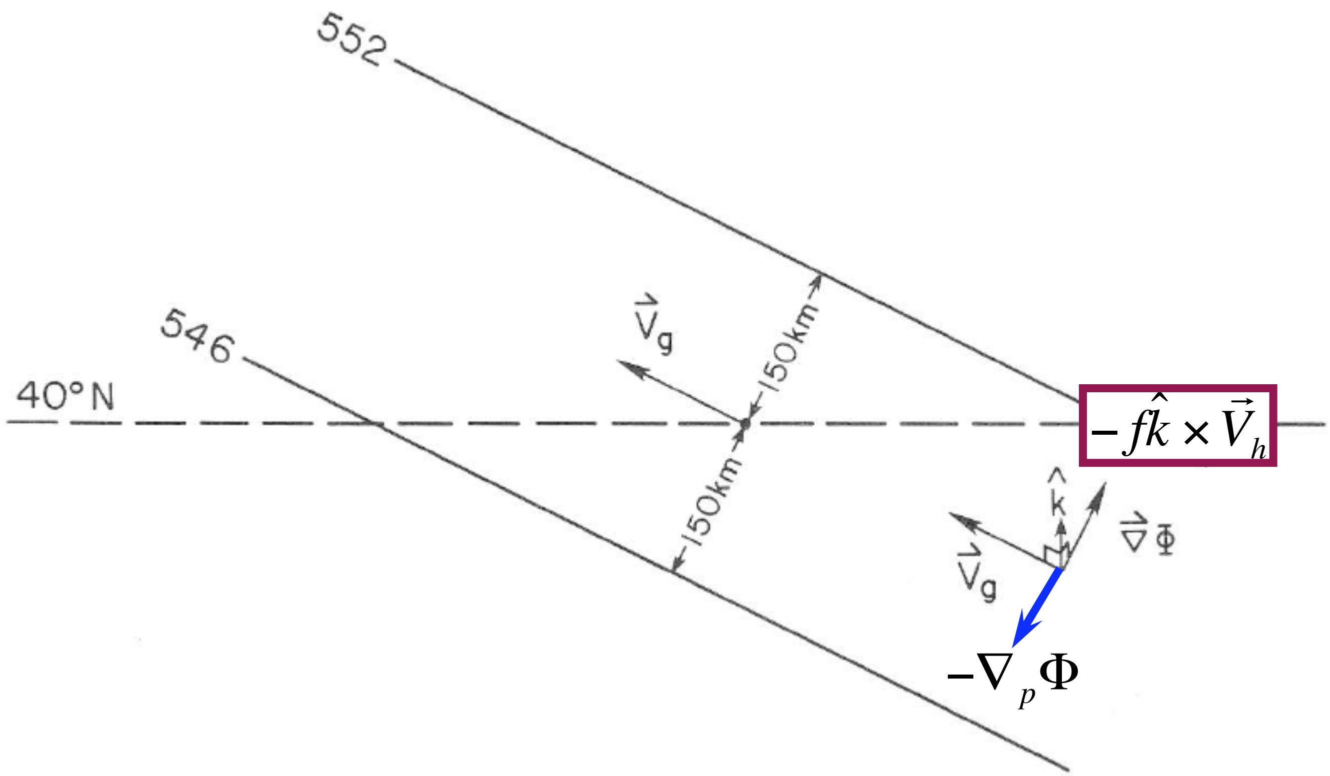
- For example, using the geopotential height as the vertical coordinate eliminates density from the pressure gradient force, and the horizontal momentum equation can be written as

$$\frac{D\vec{V}_h}{Dt} = -\vec{\nabla}_p\Phi - f\hat{k} \times \vec{V}_h$$

- **Where the Coriolis force is**

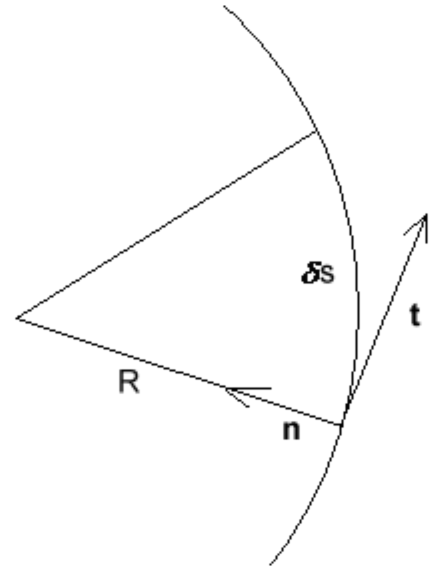
$$-f\hat{k}\times\vec{V}_h = f\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & 1\\ u & v & 0\end{vmatrix} = -f\left(-v\hat{i}+u\hat{j}\right) = 2\Omega v\sin\phi - 2\Omega u\sin\phi$$

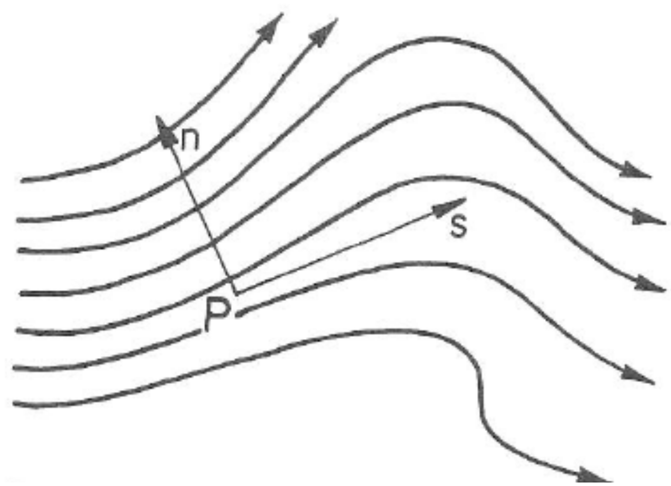
which we can confirm with the picture below.



- Another example is the use of the natural coordinate system whose orientation is determined by the direction of the flow itself.

- As seen in the figures to the right, the natural coordinate system is defined by a set of three orthogonal unit vectors, $\hat{t}$, $\hat{n}$, & $\hat{k}$, such that $\hat{t}$ is oriented parallel to the horizontal velocity at each point, $\hat{n}$ is directed normal to the velocity and positive to the left of motion, and $\hat{k}$ is positive upward.

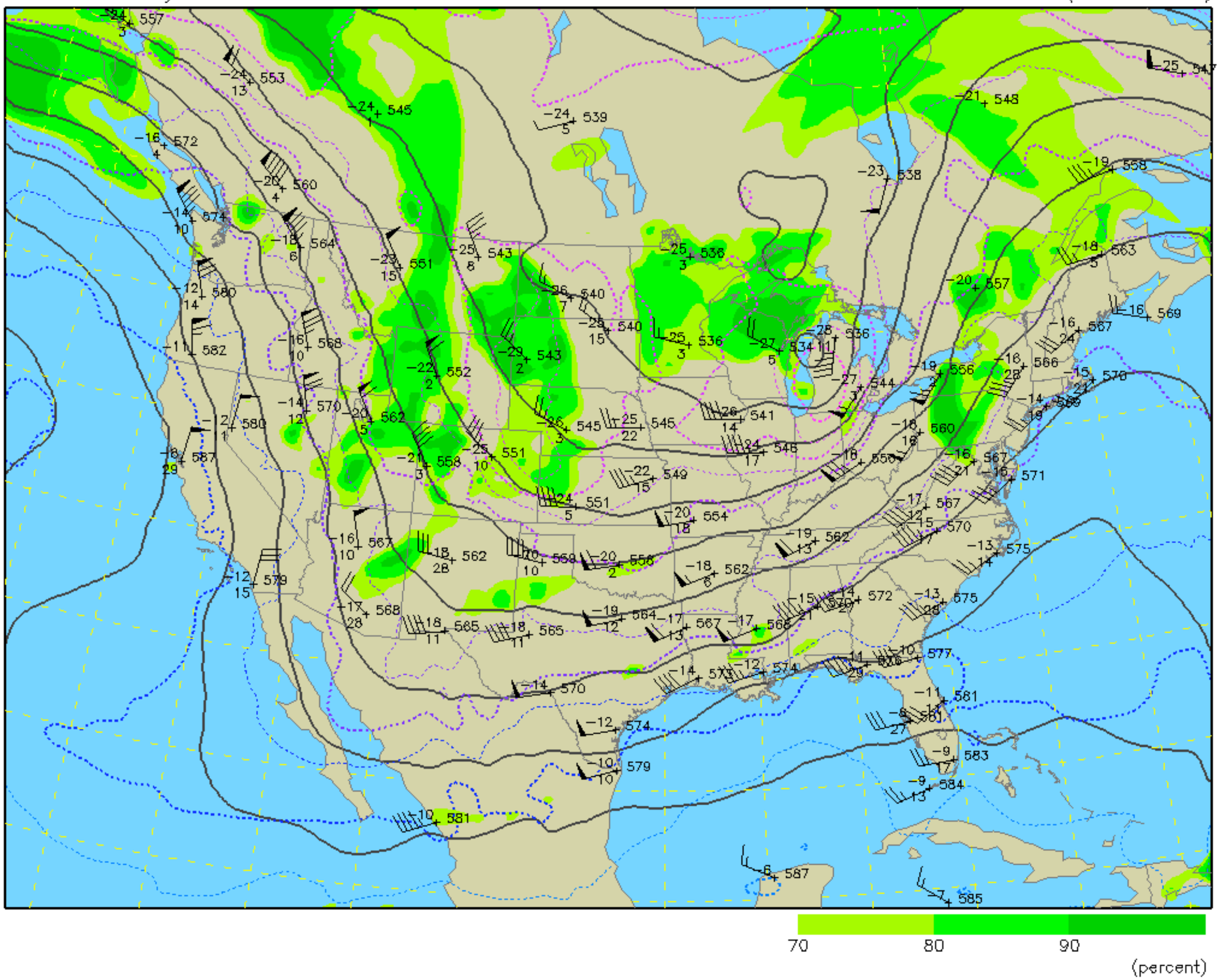


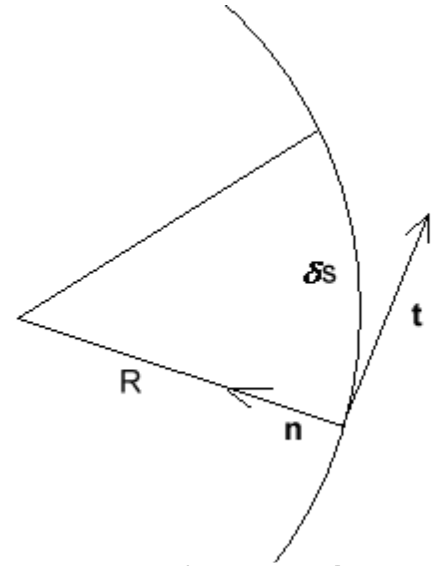


- We also define δs as the horizontal arc distance along a curve followed by an air parcel (a streamline), and R as the radius of curvature of the motion.

500 mb Heights (dm) / Temperature (°C) / Humidity (%)

0–hour analysis valid 0000 UTC Mon 15 Nov 2010 RUC (00z 15 Nov)





- R is defined as positive when the parcel turns toward the left (+$\hat{n}$ direction) following the motion, and R < 0 when the flow turns toward the right.
- In this coordinate system, $\vec{V} = V\hat{t}$ where V is the speed of the flow such that $V \equiv \frac{Ds}{Dt}$.
- The acceleration following the curved motion is therefore

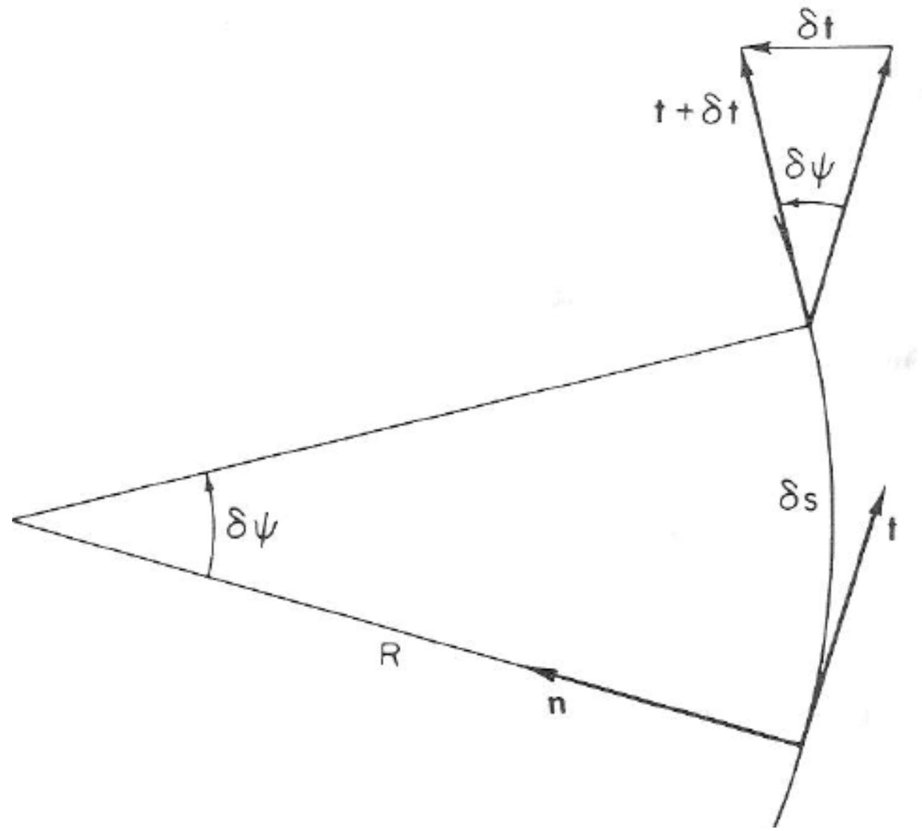
$$\frac{D\vec{V}}{Dt} = \frac{D}{Dt}\left(V\hat{t}\right) = \hat{t}\,\frac{DV}{Dt} + V\,\frac{D\hat{t}}{Dt}$$

which is the sum of the change in the wind speed in the $\hat{t}$ direction (the direction of motion) and the speed times the change in the orientation of the unit vector $\hat{t}$ following the motion.

- To determine what $\frac{D\hat{t}}{Dt}$ looks like we consult the figure on the next page which is almost identical to the $\frac{D\hat{i}}{Dt}$ image.

- **Recalling the geometric and trigonometric theorems we've been using for weeks, we see that:**

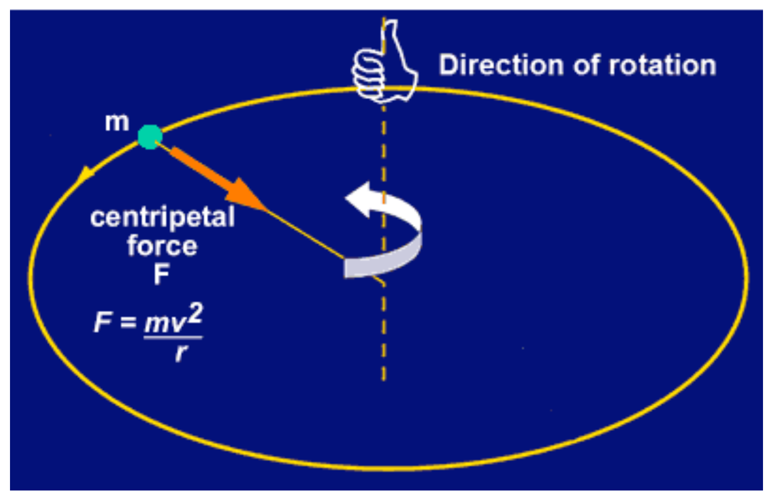
$$\left|\delta\hat{t}\right| = \left|\delta\psi\right| \cdot 1 = \left|\frac{\delta s}{R}\right| \quad \text{and} \quad \left|\frac{\delta\hat{t}}{\delta s}\right| = \left|\frac{1}{R}\right|$$



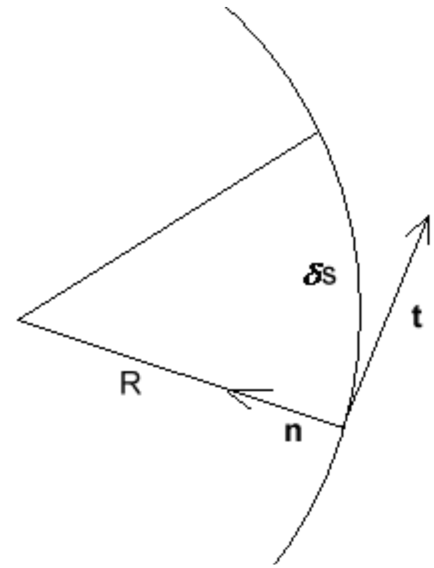
- **We also note that:**

$$\frac{D\hat{t}}{Dt} = \frac{D\hat{t}}{Ds}\frac{Ds}{Dt} \quad \text{such that} \quad \frac{D\hat{t}}{Dt} = \frac{\hat{n}}{R}V$$

- **Therefore** $\frac{D\vec{V}}{Dt} = \hat{t}\frac{DV}{Dt} + \hat{n}\frac{V^2}{R}$**, with the second term on the right hand side saying the rate of change of the unit vector** $\hat{t}$ **following the curved motion is just the centripetal force per unit mass.**



- We may now write our important force terms in natural coordinates as well.



- The Coriolis force always acts perpendicular to the direction of flow, so its natural coordinate form is simply

$$f\hat{k} \times \vec{V}_h = fV\hat{n}$$

- The pressure gradient force is $-\vec{\nabla}_p \Phi = -\left(\hat{t}\,\frac{\partial \Phi}{\partial s} + \hat{n}\,\frac{\partial \Phi}{\partial n}\right)$

  which is the sum of how the geopotential height changes along the direction of the motion and how it changes in the direction perpendicular to the flow.

- The two horizontal momentum equations in natural coordinates can then be written in component form by collecting all of the terms above in the $\hat{t}$ and $\hat{n}$ directions:

$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s} \quad \text{and} \quad \frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

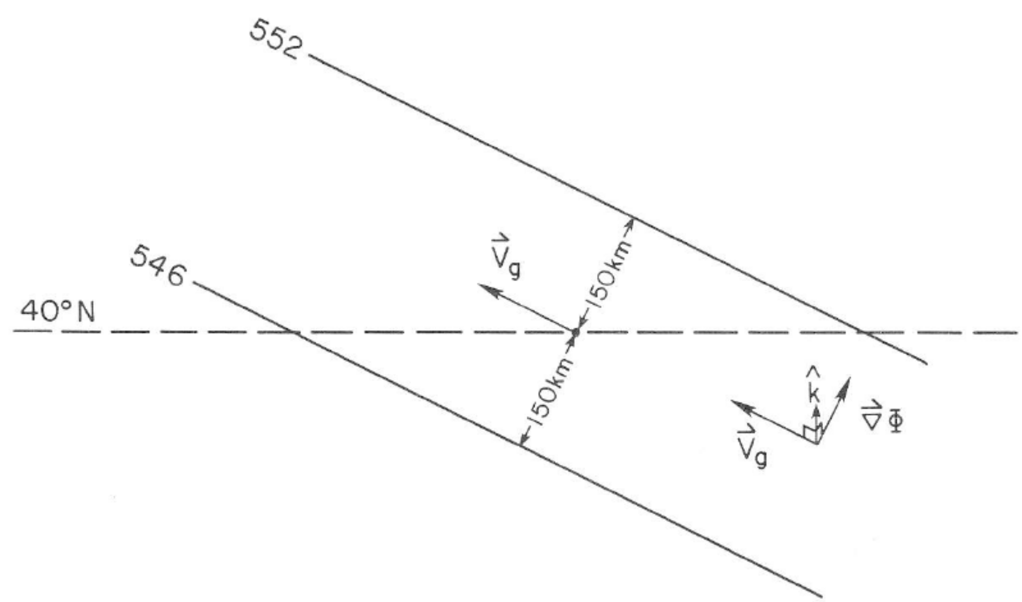
$$\frac{DV}{Dt} = -\frac{\partial \Phi}{\partial s} \quad \text{and} \quad \frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

- Thus, we have two horizontal momentum equations in natural coordinates, one for the balance of forces parallel to the flow ($\hat{s}$ or $\hat{t}$ direction) and one for the balance of forces perpendicular to the flow ($\hat{n}$ direction).
- Note that for flow parallel to the geopotential height contours $\frac{\partial \Phi}{\partial s} = 0$ and there is no acceleration of the flow.
- In such cases, we are only left with the momentum equation that represents the balance of forces perpendicular to the flow.
- We therefore have a three way balance between the centripetal / centrifugal acceleration (whether in the + or - $\hat{n}$ direction), the Coriolis force, and the pressure gradient force normal to the flow.

- We can now look at four special cases, each of which represents a different balance between the forces.
- The first special case is when the geopotential height contours are straight, $R \rightarrow \infty$ (no curvature), and we are left with a balance between the pressure gradient and Coriolis forces; i.e., it's geostrophic balance!

$$V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$$

This says that the geostrophic wind speed is dependent on the height gradient in the direction perpendicular to the flow.



- The second special case is if / when we have uniform geopotential height on a pressure surface (no isolines on a weather map!) and a balance between the Coriolis and centrifugal forces:

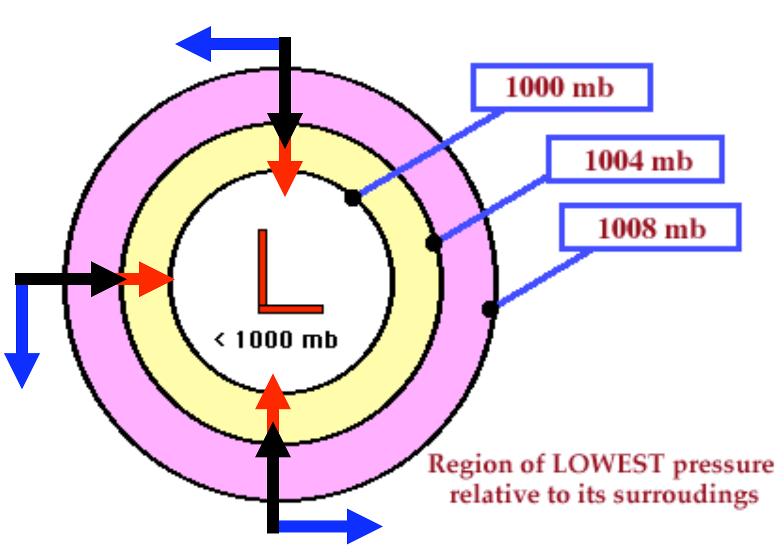
$$\frac{V^2}{R} + fV = 0$$

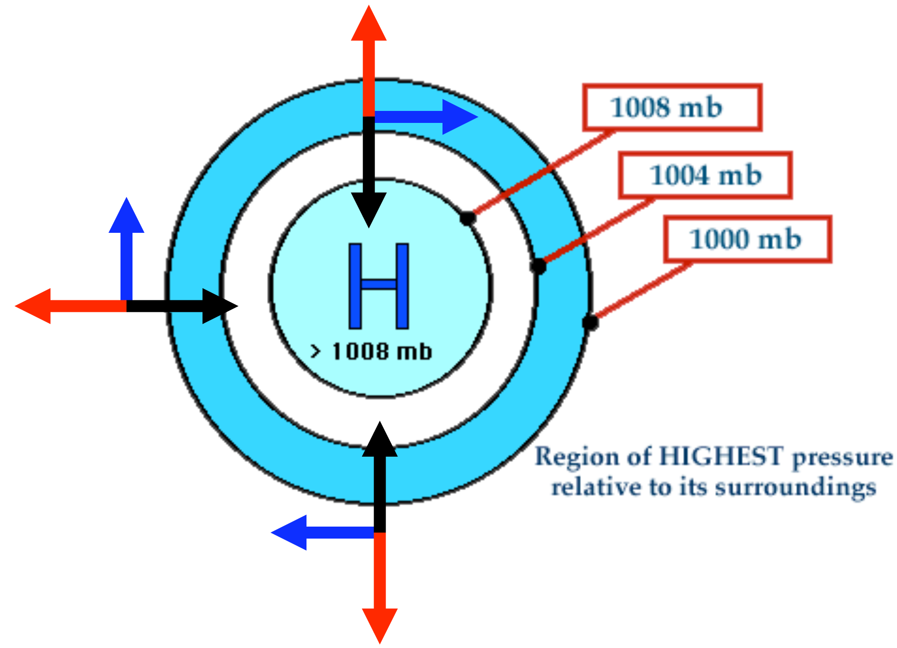
which we solve for the radius of curvature to find

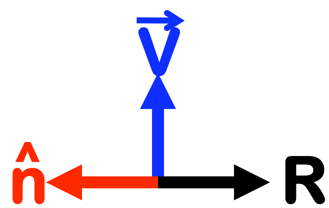
$$R = -\frac{V}{f}\hat{n}$$

- We know that there is no acceleration of the flow, so V is constant and if we neglect the small latitudinal dependence of f, R must be constant as well.

- This means we have circular motion such that R and $\hat{n}$ are pointed in the opposite direction which is only true for clockwise motion (high pressure in the Northern Hemisphere) as seen on the next slide.





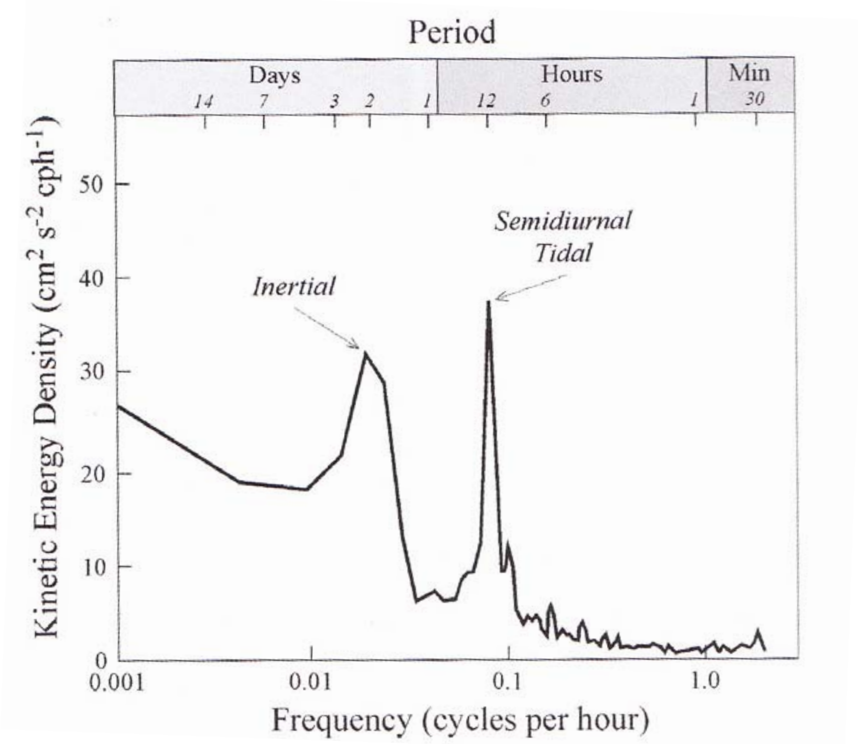


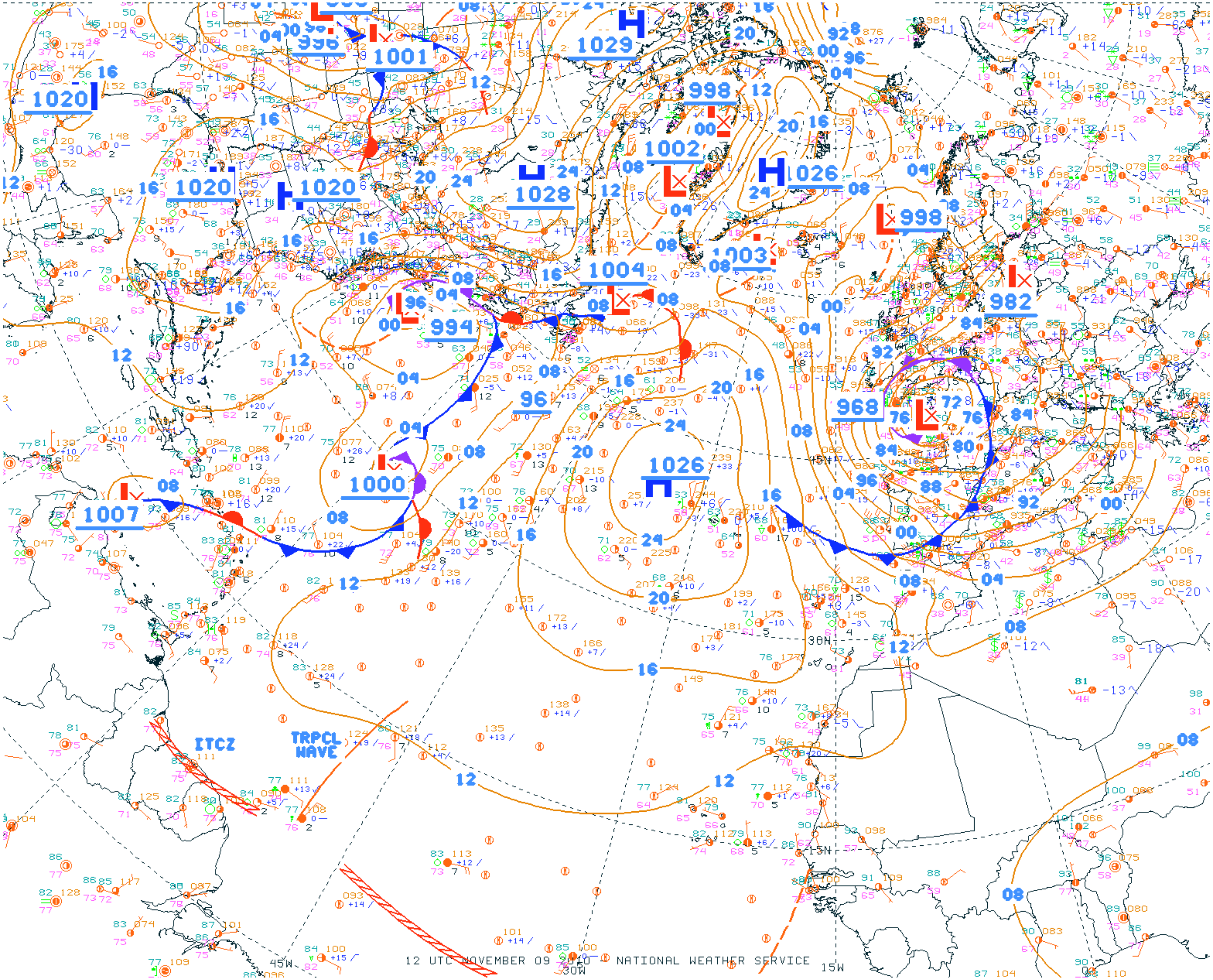
- The period of this circular motion is

$$P = \left|\frac{2\pi R}{V}\right| = \frac{2\pi}{|f|} = \frac{2\pi}{2\Omega|\sin\phi|} = \frac{\pi}{(2\pi/day)|\sin\phi|} = \frac{1/2day}{|\sin\phi|}$$

which says that the period depends on the latitude, such that there is a longer period (larger circle) closer to the equator.

- This type of flow is called inertial or an inertial oscillation (because the Coriolis and centrifugal forces are due to using a non-inertial frame) and the circle with radius R is called an inertial circle.
- In the atmosphere, horizontal motions are nearly always driven by pressure gradients, so that the conditions for inertial flow (uniform pressure) rarely exist.
- In the ocean however, currents are often generated by transient winds blowing across the surface and not internal pressure gradients.
- Therefore, inertial oscillations can exist in the ocean and are quite common as seen to the right.



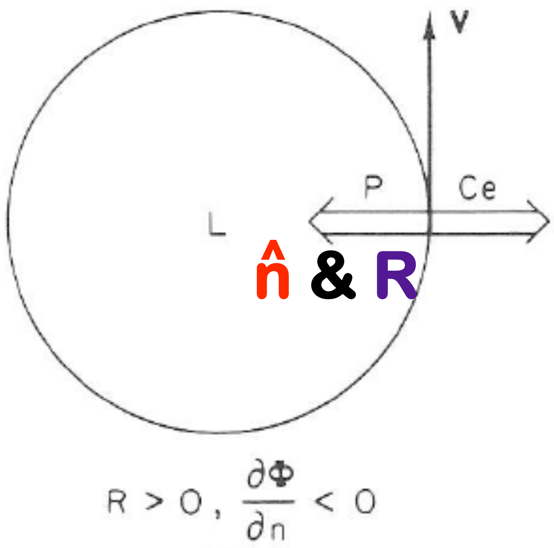

12 UTC NOVEMBER 09 2010 - NATIONAL WEATHER SERVICE
ITCZ
TRPCL WAVE
1029
1001
998
1002
1026
1020
1020
1020
1028
998
1004
1003
994
982
968
1026
1000
1007
45W
30W
15W
30N
45N
15N

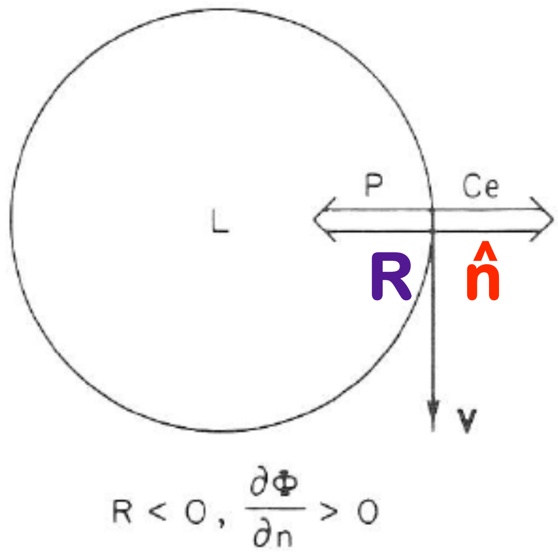
- A third special case occurs when the time and horizontal scales of a disturbance are so small that the Coriolis force may be neglected, generally < 1/2 day, like for tornadoes, water spouts and dust devils.
- In this case, called cyclostrophic balance, we are left with the pressure gradient and centrifugal forces:

$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n}$$

$$V = \sqrt{-R\frac{\partial \Phi}{\partial n}}$$

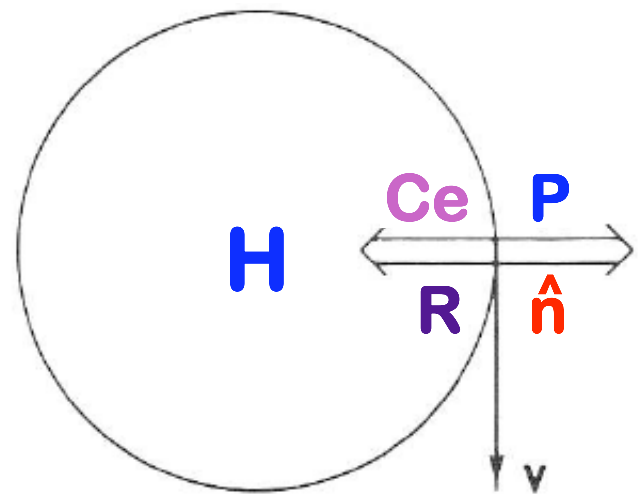
- Requiring that the quantity under the square root be positive for real solutions, we need R > 0 for dΦ/dn < 0 and R < 0 for dΦ/dn > 0 which works for the areas of low pressure seen in the figure on the next slide.





- Cyclostrophic balance works for low pressure areas!

- But what about areas of high pressure?
- As the figure indicates, R and dΦ/dn are of the same sign for normal high pressure areas and thus the quantity under the square root would be < 0 and we can't have imaginary solutions!



$R < 0, \; d\Phi / dn < 0$

- Thus, cyclostrophic flow can only exist around areas of low pressure.
- Without the constraint of the Coriolis force always acting to the right of the motion (in the Northern Hemisphere), the cyclostrophic winds around the low pressure may be counterclockwise, like around most tornadoes, or clockwise, such as with half of the observed dust devils and water spouts.
- The fourth, and final, special case and force balance in natural coordinates is deemed the gradient wind.
- In this instance, the horizontal, frictionless, steady state ($DV / Dt = 0$) flow is parallel to curved height contours, and there is a three way balance between the pressure gradient, Coriolis and centrifugal forces:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

- Like the geostrophic wind, pure gradient flow can only exist under rare conditions, but it is always possible to define a gradient wind vector.
- And since the gradient wind takes into account curved flow (like the real troughs and ridges in the atmosphere), it often is a better approximation to the real wind than the geostrophic wind.
- The gradient wind equation is

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

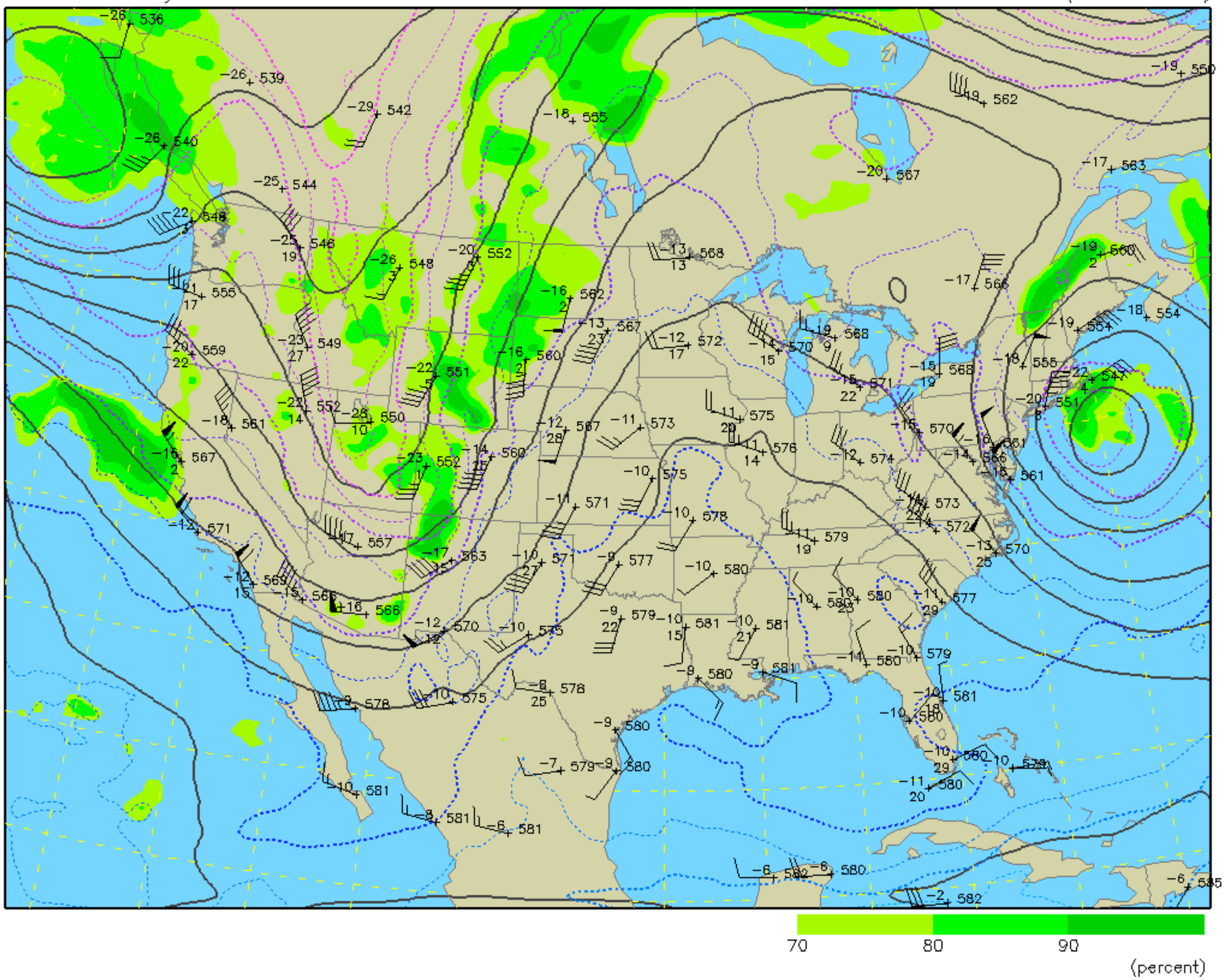
and solving for V using the quadratic formula we have:

$$V = -\frac{fR}{2} \pm \sqrt{\frac{f^2 R^2}{4} - R\frac{\partial \Phi}{\partial n}}$$

- As we have a square root, some of the mathematically possible solutions are physically impossible because V is required to be real and non-negative.

500 mb Heights (dm) / Temperature (^{o}C) / Humidity (%)

0–hour analysis valid 1200 UTC Tue 09 Nov 2010 RUC (12z 09 Nov)



- **Because we can have both positive and negative R and dΦ/dn, and there is the ± in front of the square root, there are 8 possible gradient wind solutions summarized in the table below, of which, only 4 are physically possible.**

Table 3.1 *Classification of Roots of the Gradient Wind Equation in the Northern Hemisphere*[a]

Sign $\partial\Phi/\partial n$	$R>0$	$R<0$
Positive	Positive root: unphysical	Positive root: antibaric flow (anomalous low)
	Negative root: unphysical	Negative root: unphysical
Negative	Positive root: cyclonic flow (regular low)	Positive root: ($V>-fR/2$): anticyclonic flow (anomalous high)
	Negative root: unphysical	Negative root: ($V<-fR/2$): anticyclonic flow (regular high)

- **Examining the unphysical solutions first…**

- We can see that both the positive and negative roots of the instances where R > 0 and dΦ/dn > 0 (first two lines of the left hand column of the table) are unphysical by going back to the gradient wind equation:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

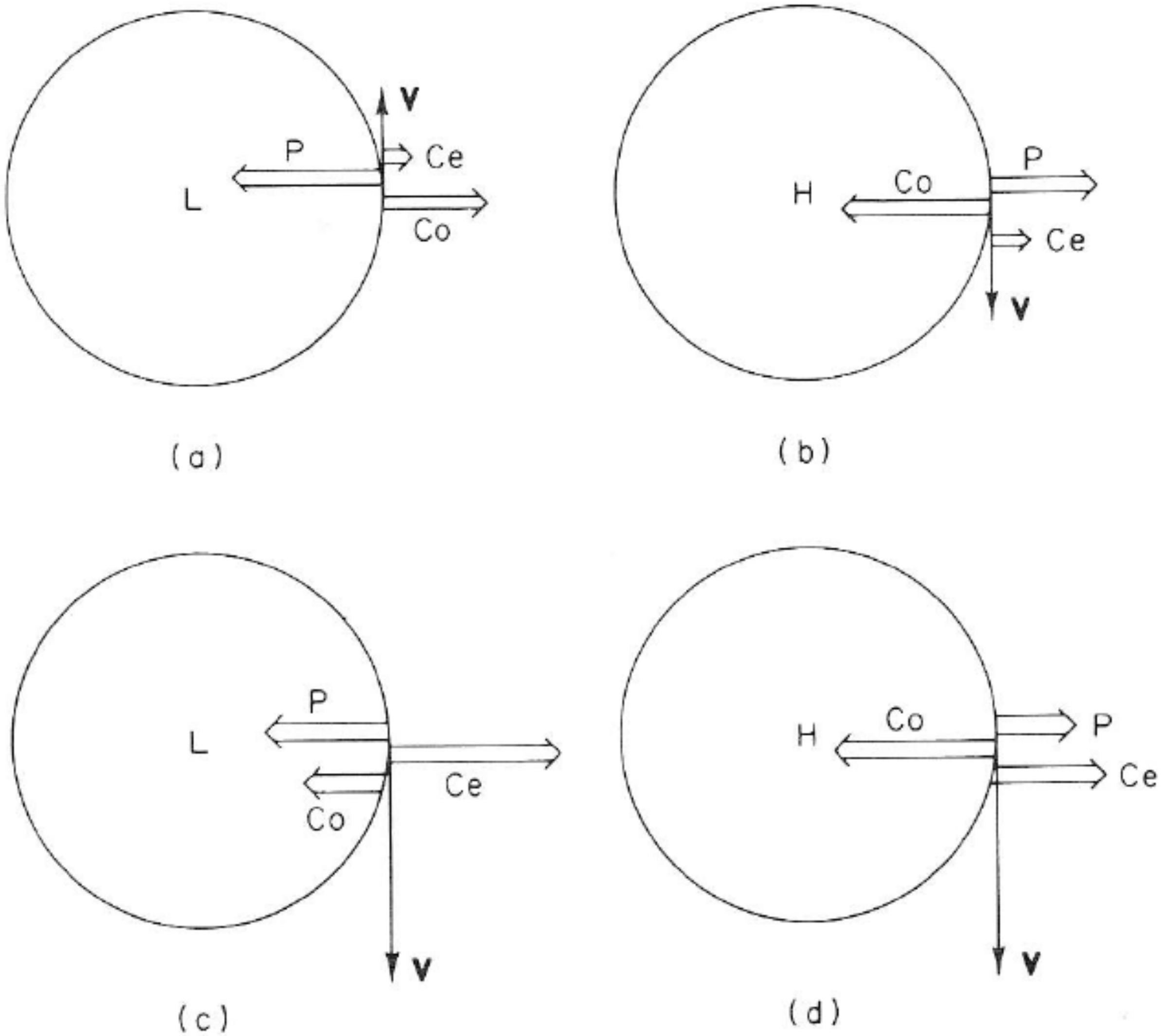
with R > 0, the left hand side is > 0 and thus dΦ/dn must be negative. Thus, any instance with dΦ/dn > 0 is not possible.

- The other two physically unrealistic cases (the negative roots of R > 0 with dΦ/dn < 0 and R < 0 with dΦ/dn > 0, respectively) are a bit more involved, but you should prove their unphysical nature to yourself (and check your reasoning with Martin's book).

- So, what about the four physically possible roots?

$R > 0$, $d\Phi / dn < 0$

$R < 0$, $d\Phi / dn < 0$, - root

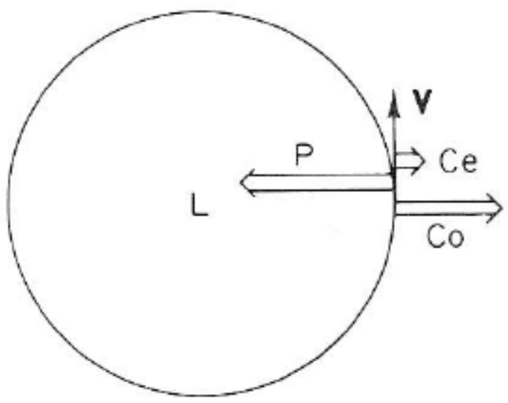


$R < 0$, $d\Phi / dn > 0$

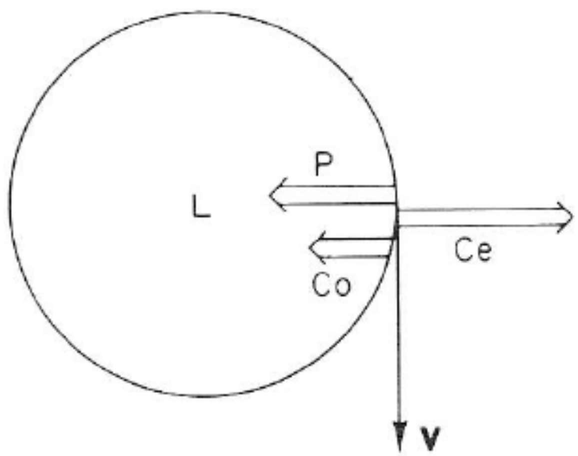
$R < 0$, $d\Phi / dn < 0$, + root

- As can be seen at the left, there are two "normal" flow cases, one high (b) and one low pressure area (a), and two "anomalous" flow cases, again one area of high (d) and one area of low pressure (c).

R > 0, $d\Phi$ / dn < 0



(a)



(c)

R < 0, $d\Phi$ / dn > 0

- The regular low has counterclockwise flow and the pressure gradient force balancing both the centrifugal and Coriolis forces:

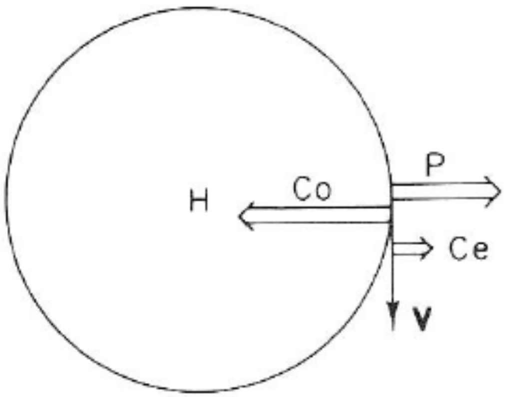
$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n}$$

- The anomalous low has clockwise flow with the centrifugal force balancing both the pressure gradient and Coriolis forces:

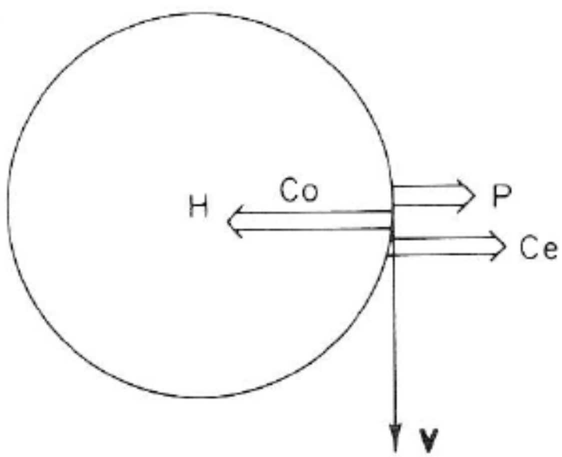
$$\frac{V^2}{R} = -\frac{\partial \Phi}{\partial n} - fV$$

which looks weird and doesn't actually happen in the Northern Hemisphere, but is physically possible.

$R < 0$, $d\Phi / dn < 0$,
\- root



(b)



(d)

$R < 0$, $d\Phi / dn < 0$,
\+ root

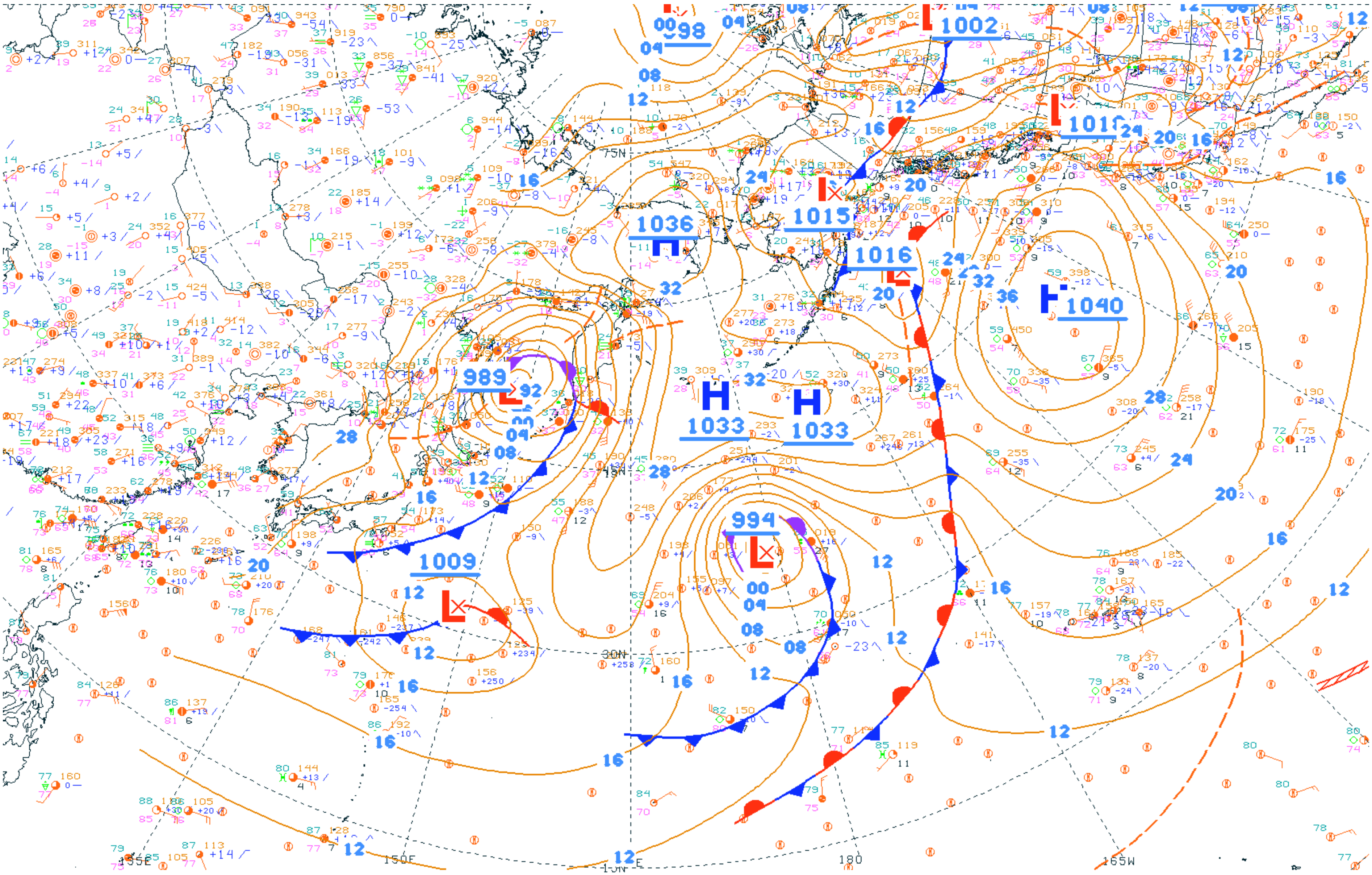
- The flow around both of the physically realistic high pressure areas is clockwise as we would expect, but because both R and dΦ / dn are negative, we must watch the sign of the quantity under the square root in the gradient wind equation:

$$V = -\frac{fR}{2} \pm \sqrt{\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n}}$$

$$\frac{f^2R^2}{4} - R\frac{\partial\Phi}{\partial n} > 0$$

$$\frac{f^2R^2}{4} > \left|R\frac{\partial\Phi}{\partial n}\right| \longrightarrow \frac{f^2|R|}{4} > \left|\frac{\partial\Phi}{\partial n}\right|$$

- From this equation, we see that as $R \rightarrow 0$, i.e., we approach the center of the high, the pressure field must go flat (no gradient) and thus we have calmer winds in the center of the high!



00 UTC NOVEMBER 15 2010 - NATIONAL WEATHER SERVICE

- The definition of the geostrophic wind $V_g = -\frac{1}{f}\frac{\partial \Phi}{\partial n}$ can be used to rewrite the gradient wind equation such that:

$$\frac{V^2}{R} + fV = -\frac{\partial \Phi}{\partial n} \quad \text{becomes} \quad \frac{V^2}{R} + fV - fV_g = 0$$

- Dividing through by f V shows that the ratio of the geostrophic to the gradient wind is

$$\boxed{\frac{V_g}{V} = 1 + \frac{V}{fR}}$$

- Thus, for normal cyclonic flow (counterclockwise around a cyclone, a low pressure area in the Northern Hemisphere), R > 0, the Coriolis and centrifugal forces are in the same direction and the geostrophic wind is larger than the gradient wind.

- For normal anticyclonic flow (clockwise around a high pressure area in the Northern Hemisphere), R < 0, the Coriolis and centrifugal forces oppose one another, and the geostrophic wind is smaller than the gradient wind.

- Thus, in regions of cyclonic curvature (R > 0; troughs) the geostrophic wind is an overestimate of the balanced wind, and in regions of anticyclonic curvature (R < 0; ridges) the geostrophic wind is an underestimate.

- The difference between the gradient and geostrophic winds is usually < 15 % in the midlatitudes, but in the tropics, where f is small, V / f R (which scales just like the Rossby number!) can be large, and the gradient wind should be used.