The Continuity Equation

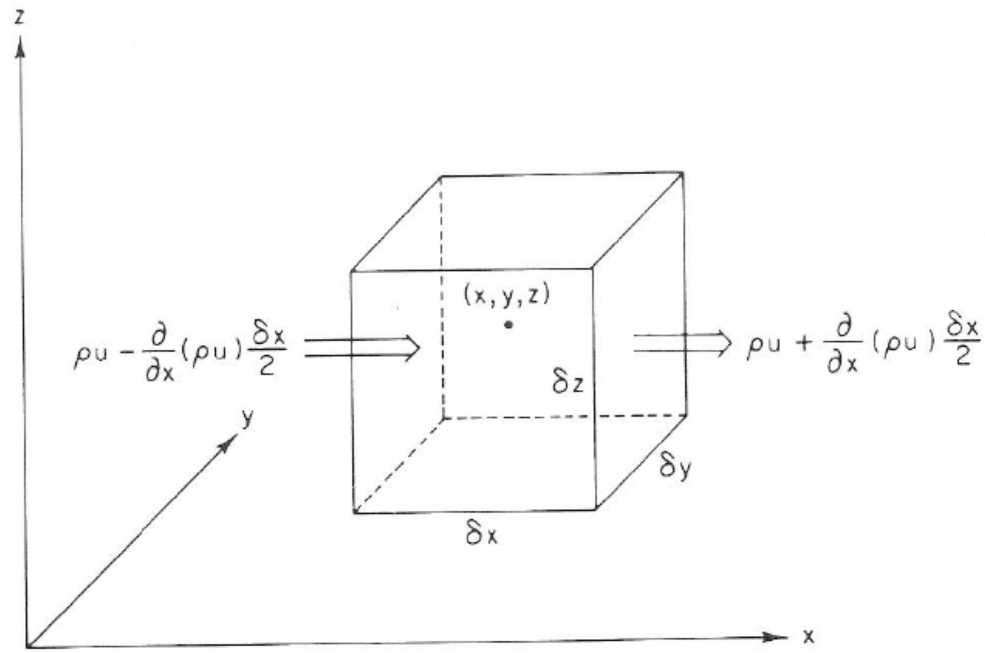
- Over the last few classes, we have derived the first of the basic conservation laws of fluid dynamics, the momentum equation, in both its three-dimensional vector and component forms.

- We've also already derived the second conservation law, the thermodynamic energy equation (a.k.a. The First Law of Thermodynamics), and thus we will skip over the last section of Chapter 2 in Holton, but you are still responsible for the information therein!

- Today we will tackle the final conservation law, the conservation of mass, which is expressed mathematically by the continuity equation, which can be written in both the Eulerian (fixed) and Lagrangian (go with the flow) frames of reference.

- Consider the fixed in space (Eulerian) cube in the figure below with volume δx δy δz.

- For this fixed cube, the net rate of mass inflow/outflow through the sides must equal the accumulation of mass within the volume.



- If we consider the rate of mass inflow per unit area (mass flux) at the center of the box in the x-direction to be ρu, then the rate of mass inflow through the left hand face is

$$\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$$

$$\delta M = \rho \delta x \delta y \delta z$$

$$\frac{\delta M}{\delta t} = \frac{\rho \delta x \delta y \delta z}{\delta t}$$

per unit area of the x - face ($\delta y \delta z$)

$$\frac{\delta M}{\delta t} = \frac{\rho \delta x \delta y \delta z}{\delta t \delta y \delta z} = \rho \frac{\delta x}{\delta t} = \rho u$$

- And the rate of mass outflow per unit area through the right hand face is

$$\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}$$

- The net rate of mass flow into the cube in the x-direction is therefore (eliminating the per unit area constraint):

$$\left(\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right)\delta y \delta z - \left(\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right)\delta y \delta z = -\frac{\partial}{\partial x}(\rho u)\delta x \delta y \delta z$$

- With similar expressions for the y and z faces of the cube, we may write the net rate of mass flow over all three faces as

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x \delta y \delta z$$

which, per unit volume ($\delta x\ \delta y\ \delta z$), reduces to just the term in brackets, which can be written as $-\vec{\nabla} \bullet \rho \vec{V}$.

- This net mass flow per unit volume must equal the rate of increase of mass per unit volume in the cube:

$$-\vec{\nabla} \bullet \rho\vec{V} = \frac{\partial \rho}{\partial t}, \text{ with } \frac{\partial M}{\partial t} = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z \text{ and the size of the box is fixed.}$$

OR

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \bullet \rho\vec{V} = 0}$$

- This is the mass divergence or flux form of the continuity equation which says that the mass in a volume can only change locally (Eulerian frame) through the flux convergence (more mass coming in than going out) or divergence (more mass going out than coming in) of mass into / out of the volume.

- An alternate form of the equation (Lagrangian frame) can be obtained by rewriting the expression on the previous slide using the generic Eulerian to Lagrangian transform applied to the mass per unit volume (ρ):

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \vec{V} \bullet \vec{\nabla}\rho$$

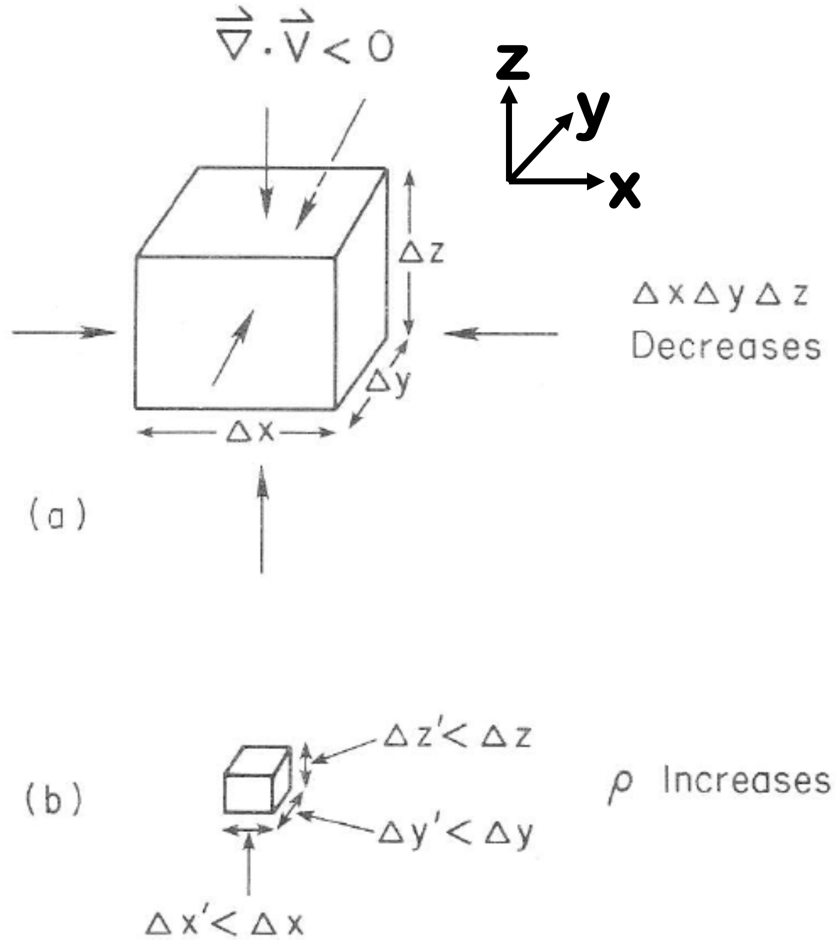
and the vector identity $\vec{\nabla} \bullet \left(a\vec{b}\right) = a\vec{\nabla} \bullet \vec{b} + \vec{b} \bullet \vec{\nabla} a$, so that

$$\frac{D\rho}{Dt} - \vec{V} \bullet \vec{\nabla}\rho + \rho\vec{\nabla} \bullet \vec{V} + \vec{V} \bullet \vec{\nabla}\rho = 0$$

$$\boxed{\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \bullet \vec{V} = 0}$$

- This is the velocity divergence form of the continuity equation that states the fractional rate of change of mass per unit volume following the motion is equal to the negative (opposite sign) of the divergence of the velocity.

- Physically, this form of the continuity equation says that the fractional rate of change of mass in a volume can only be changed through three-dimensional convergence or divergence.



- Consider the figure to the left and this re-expression of the equation:

$$\frac{1}{\rho}\frac{D\rho}{Dt}=-\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

- In the figure, there is convergence, for which $\partial u/\partial x<0$ (in the + x-direction the winds go from westerly to easterly), $\partial v/\partial y<0$ (southerly to northerly in the + y-direction), and $\partial w/\partial z<0$ (upward to downward motion).

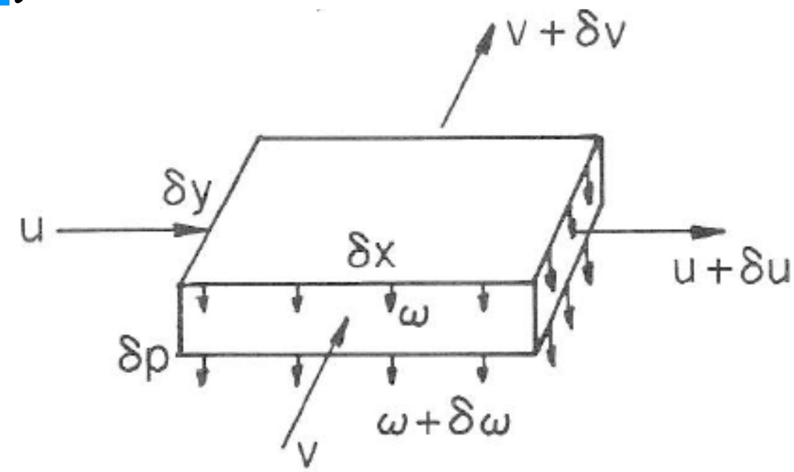
- Thus, for convergence, $\vec{\nabla} \bullet \vec{V} < 0$ (the wind field is squashing the volume), the parcel will shrink in size and the density of the parcel will increase because

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\vec{\nabla} \bullet \vec{V}$$

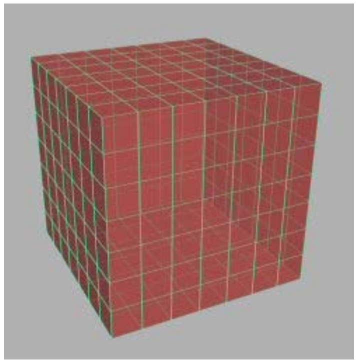
- To illustrate further why this form of the equation is so useful and why the divergence of the velocity field is so crucial to atmospheric dynamics, we consider the air parcel in the figure below with volume, δx δy δp.

- In the hydrostatic atmosphere, the mass of the block is

$$\delta M = \rho \delta x \delta y \delta z = -\frac{\delta x \delta y \delta p}{g}$$



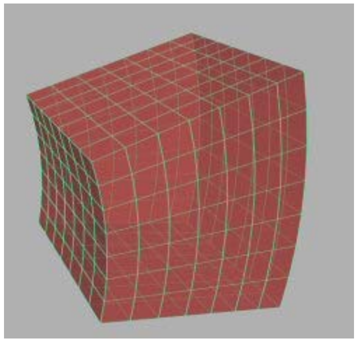
- If the mass of the cube is not changing with time:



$$\frac{D}{Dt}(\delta x \delta y \delta p) = 0$$

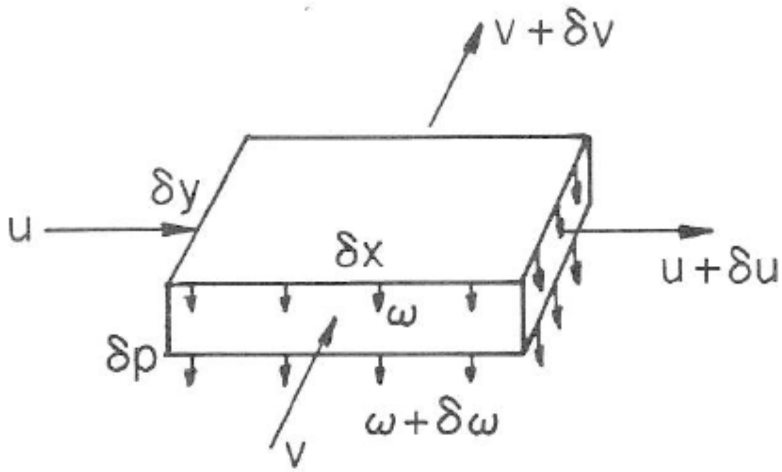
OR

$$\delta y \delta p \frac{D}{Dt}(\delta x) + \delta x \delta p \frac{D}{Dt}(\delta y) + \delta x \delta y \frac{D}{Dt}(\delta p) = 0$$



- With the passage of time, the block will be twisted and distorted by the stresses and deformations in the motion field, but we are only concerned with the rates of change of the three dimensions (δx, δy, δp) of the parcel at the initial moment when it's an approximate cube.

- Based on the figure, the time rate of change of the zonal dimension, δx, is equal to δu, the difference in the zonal velocities averaged over the two δy δp faces of the cube.



- **If the dimensions of the box are very small compared to the space scale of the velocity fluctuations, we may write:**

$$\frac{D}{Dt}(\delta x) = \delta u = \frac{\partial u}{\partial x}\delta x$$

- **Expressing the time rates of change of δy and δp in the same manner, we may write the conservation of mass as**

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}\right)\delta x \delta y \delta p = 0$$

Diving through by the unit volume, we have the continuity equation in pressure coordinates:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \quad \text{OR} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \vec{\nabla} \bullet \vec{V}_h = -\frac{\partial \omega}{\partial p}$$

which says that the divergence of the horizontal wind is related to the vertical profile of vertical motion (ω).

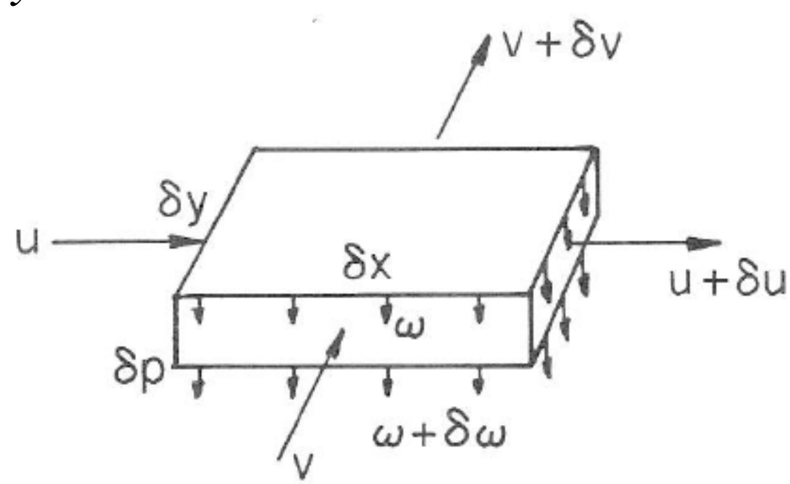
- This expression is equivalent to the Lagrangian velocity divergence equation derived earlier, but here we have assumed the mass of the parcel between two pressure levels cannot change, i.e.

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \bullet \vec{V} = 0$$

$$\frac{DM}{Dt} = \frac{1}{\rho}\frac{D\rho}{Dt} = 0$$

which is equivalent to saying the atmosphere is incompressible.

- In the block example, $\partial u / \partial x$ and $\partial v / \partial y$ are > 0 (mass is evacuated from the box), so $-\partial\omega/\partial p > 0$ or $\partial\omega > 0$ as we go in the –p direction (up), so that $\omega = \frac{\partial p}{\partial t} > 0$ and we have sinking air across the volume assuming no vertical motion to begin with.



- As another example, consider the figure below and rewrite the expanded conservation of mass equation:

$$\delta y \delta p \frac{D}{Dt}(\delta x) + \delta x \delta p \frac{D}{Dt}(\delta y) + \delta x \delta y \frac{D}{Dt}(\delta p) = 0$$

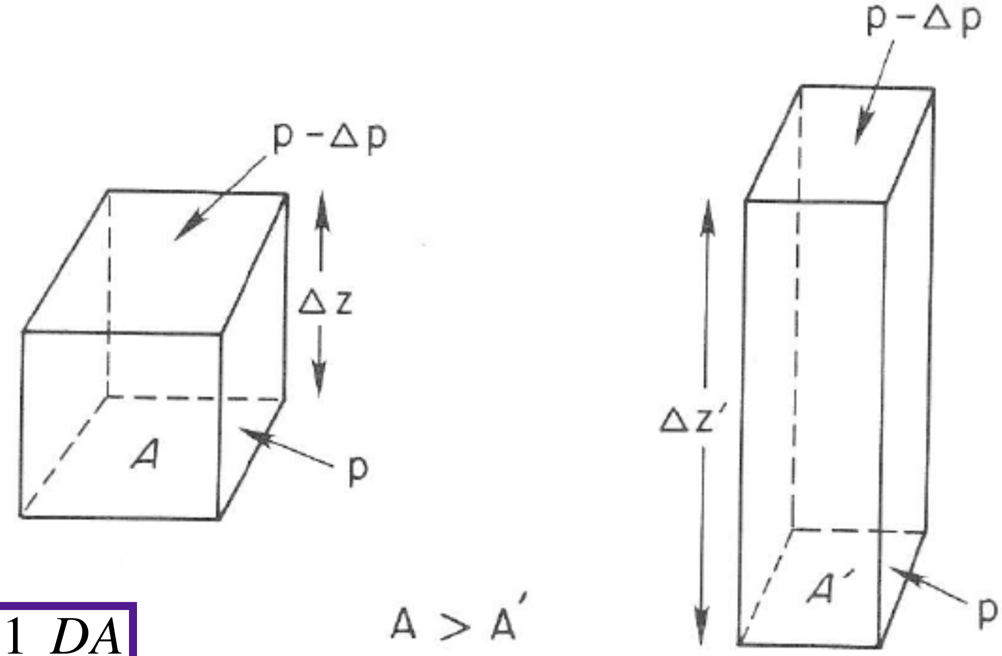
as

$$\delta p \frac{DA}{Dt} + A \frac{D}{Dt}(\delta p) = 0$$

with A = δx δy.

- **Substituting for the time rate of change of δp as before,** $\frac{D}{Dt}(\delta p) = \delta\omega = \frac{\partial \omega}{\partial p}\delta p$**,**

and dividing through by A δp (the volume) we have

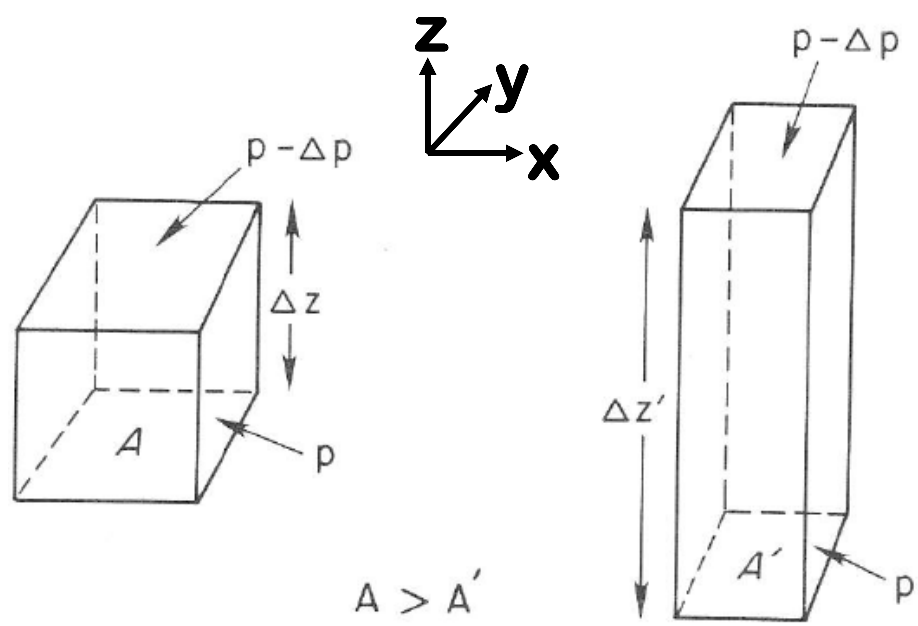


$$\boxed{\frac{1}{A}\frac{DA}{Dt} + \frac{\partial \omega}{\partial p} = 0}$$

OR

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{1}{A}\frac{DA}{Dt}}$$

- Applying this to the figure, if there is horizontal convergence, $\partial u/\partial x + \partial v/\partial y < 0$, the area of the surface the convergence is affecting must shrink, $DA/Dt < 0;\ A' < A$.
- Since the total mass contained between the two pressure surfaces in the volume must remain constant, the column must expand vertically to compensate for the decrease in horizontal area of the top and bottom faces!
- Which is to say that $-\partial \omega/\partial p < 0$ or $\partial \omega < 0$ as we go in the – p-direction (up) so that $\omega = \dfrac{\partial p}{\partial t} < 0$ and we have rising air in the volume assuming no vertical motion to begin with or at the bottom and top boundaries of the cube.



- Thus, an atmospheric volume (really a column) behaves like it were incompressible: If it is squeezed around its middle, air squirts upward and downward like a tube of toothpaste!

- The preceding derivation and discussion proved that horizontal divergence or convergence causes vertical motion in a column, and thus vertically integrating the continuity equation can give us an estimate of the expected vertical motion.

- This method of estimating the vertical velocity (which is 3 orders of magnitude smaller than the horizontal velocity for synoptic scale motions and thus difficult to measure directly) is called the kinematic method.

- We integrate the continuity equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=-\frac{\partial \omega}{\partial p}$, from a reference pressure level p_s to any level p, to find:

$$\omega(p)=\omega(p_s)-\int_{p_s}^{p}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)_p dp=\omega(p_s)+(p_s-p)\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right)$$

where the < > brackets denote a pressure-weighted vertical average:

$$\langle\ \rangle=\frac{1}{p-p_s}\int_{p_s}^{p}(\ )dp$$

- To good approximation in the atmosphere $\omega\approx-\rho g w$ (see Holton section 3.5 for proof) and using the hydrostatic equation we may write:

$$w(z)=\frac{\rho(z_s)}{\rho(z)}w(z_s)-\frac{(p_s-p)}{\rho(z)g}\left(\frac{\partial\langle u\rangle}{\partial x}+\frac{\partial\langle v\rangle}{\partial y}\right)$$

- Application of this formula to the real world requires a knowledge of the horizontal divergence.
- To determine the partial derivatives $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$, we generally use finite difference approximations.
- For example, to determine the divergence at point (x_0,y_0) in the figure below, we write

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\approx\frac{u(x_0+d)-u(x_0-d)}{2d}+\frac{v(y_0+d)-v(y_0-d)}{2d}$$

- The kinematic approximation to the vertical motion is good for a first guess estimation, but as we shall soon see, $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are nearly equal and oppositely signed in the atmosphere (it's quasi-horizontally non-divergent!) and thus small errors in the wind field lead to large errors in w.