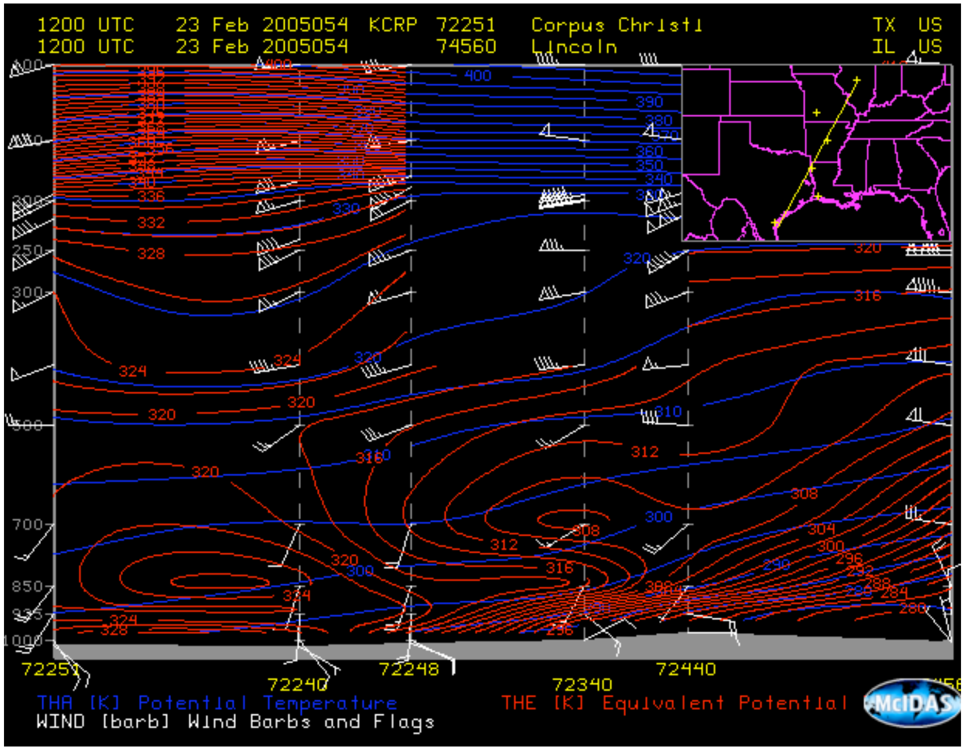
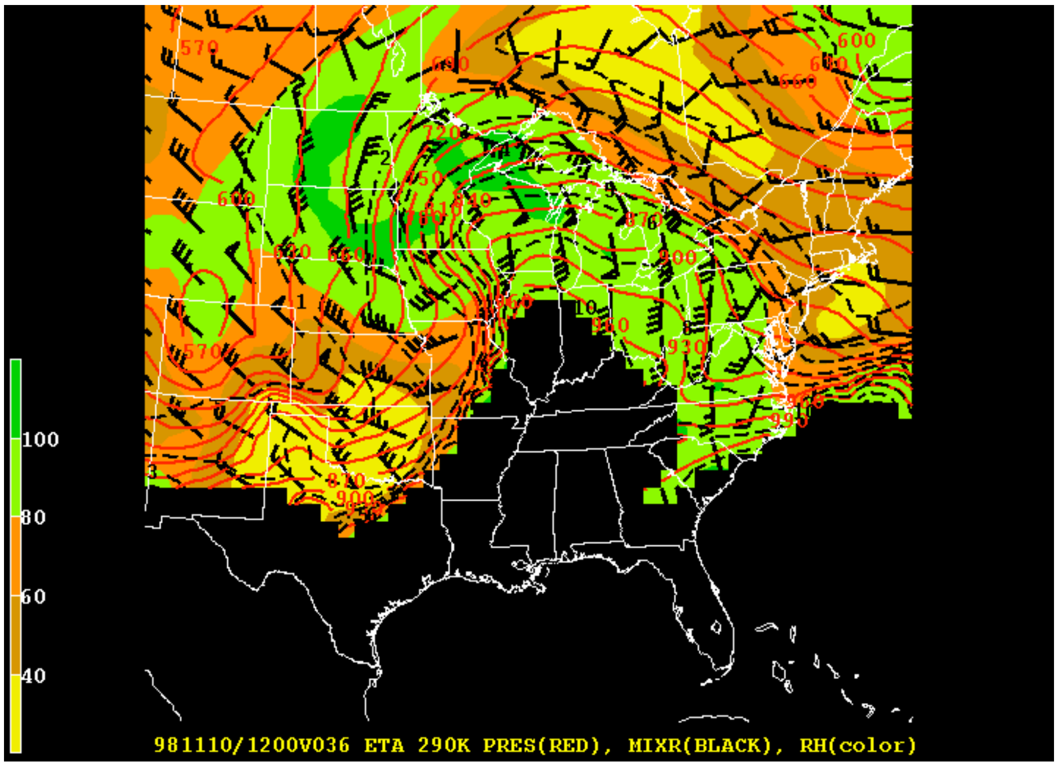
Potential Vorticity

- If we consider atmospheric flow on surfaces of constant potential temperature (θ), instead of on constant height or pressure surfaces, a remarkable and useful relationship between the rotation of a column of fluid and its depth is achieved.

- As an aside, it is desirable to choose θ as the vertical coordinate since for stable stratifications, θ is a monotonic function of height and for adiabatic processes air parcels must conserve their potential temperature.



- This second point is especially useful because it means that for adiabatic flow, an isentrope (line of constant θ) is a material surface along which parcels must remain.



- Using this constraint and applying it to a map of pressure on an isentropic surface is a handy way to diagnose vertical motion!

- If parcels do not conserve their θ, diabatic processes (e.g., latent heat release from convection) or friction must be in play: ★ The University of Utah Weather Center ★

- But, I digress! Back to potential vorticity!

- Recall, the definition of potential temperature:

$$\Theta = T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}}$$

and substitute in for temperature from the Ideal Gas Law:

$$\Theta = \frac{p}{\rho R_d}\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_p}} \quad \text{or} \quad \rho R_d \Theta = p_0^{\frac{R_d}{c_p}} p^{1-\frac{R_d}{c_p}}$$

- Using $R_d + c_v = c_p$, we may rewrite the above as:

$$\rho = \frac{p_0^{\frac{R_d}{c_p}} p^{\frac{c_v}{c_p}}}{R_d \Theta}$$

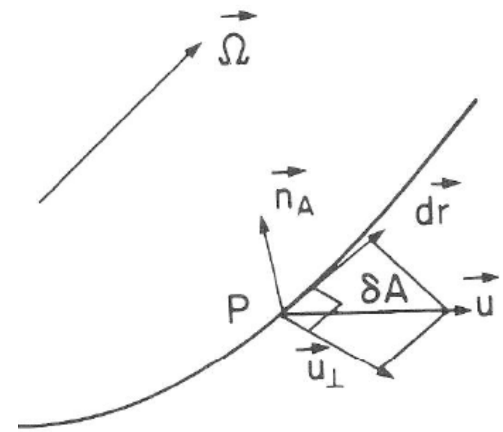
which tells us that flow on isentropic surfaces (constant θ) is barotropic, since density is only a function of pressure.

- Therefore, on isentropic surfaces there is no pressure gradient / baroclinic / solenoid term in the Circulation Theorem and the expression can be reduced to:

$$\frac{DC}{Dt} = -\oint\left(2\vec{\Omega}\times\vec{V}\right)\bullet d\vec{r} = -2\Omega\sin\phi\frac{dA}{dt}$$

or

$$\frac{d}{dt}\left(C + 2\Omega\sin\phi A\right) = 0$$



- As we previously defined, $\zeta = \frac{C}{A}$,which we can use to rewrite the above as:

$$\frac{d}{dt}\left(\zeta A + 2\Omega\sin\phi A\right) = \frac{d}{dt}\left[\left(\zeta_\Theta + f\right)A\right] = 0$$

which says that the product of the absolute vorticity and the area of the closed circulation loop is constant for adiabatic flow on an isentropic surface.

- This is great, but circulation area is not something we routinely measure, so we aim to re-express it starting with the Hydrostatic Equation:

$$-\delta p = \rho g \delta z = \frac{\delta M}{V} g \delta z = \frac{\delta M}{A} g$$

which says that the amount of mass between two θ surfaces is a function of the isobaric depth, $-\delta p$, of the column.

- Mass continuity demands that this amount of mass be conserved following the flow. We may thus solve for A:

$$A = -g \frac{\delta M}{\delta p}$$

- Alas, this expression is untenable as well because we do not routinely measure the mass of air columns. Thus, we wish to rewrite $\delta M / \delta p$.

- We rewrite the expression using the Chain Rule as:

$$A = -g\left(\frac{\delta M}{\delta\Theta}\frac{\delta\Theta}{\delta p}\right)$$

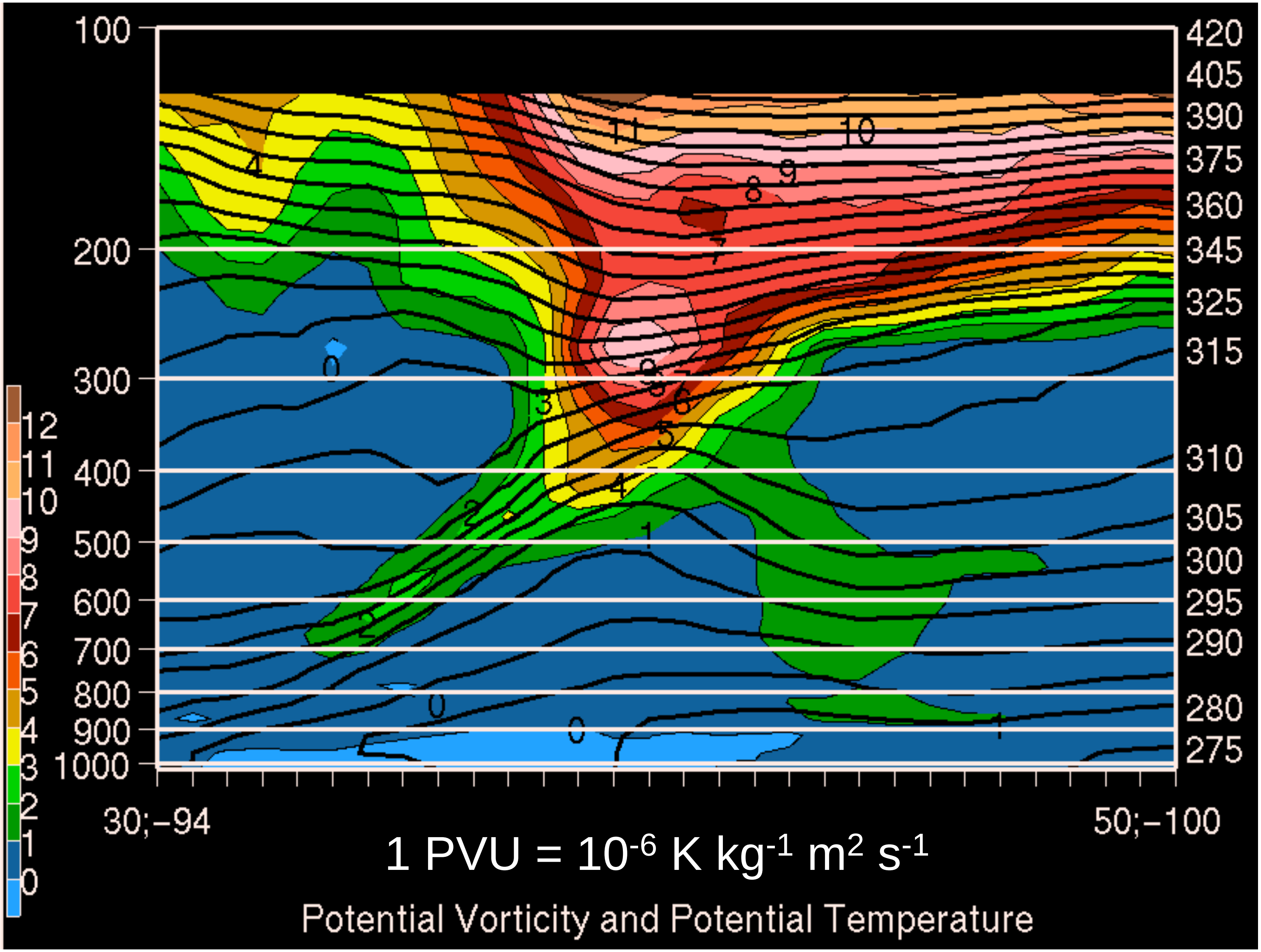
- Since both δM and $\delta\Theta$ are conserved for adiabatic flow, their ratio is a constant such that:

$$A = -g\left(\frac{\delta\Theta}{\delta p}\right)\times \text{constant}$$

- Taking the limit as $\delta\Theta, \delta p \rightarrow 0$ and using $\frac{d}{dt}\left[\left(\zeta_\Theta + f\right)A\right] = 0$ or $\left(\zeta_\Theta + f\right)A = \text{constant}$:

$$\left(\zeta_\Theta + f\right)\left(-g\frac{\partial\Theta}{\partial p}\right) = \text{constant}$$

- The left hand side of the above (the ratio of the absolute vorticity to the depth of the column) is known as the potential vorticity (PV, P, Π) and is conserved for parcels in frictionless, adiabatic flow. ★


100
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390
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345
325
315
310
305
300
295
290
280
275
12
11
10
9
8
7
6
5
4
3
2
1
0
30;−94
50;−100
1 PVU = 10^{-6} K kg^{-1} m^2 s^{-1}
Potential Vorticity and Potential Temperature

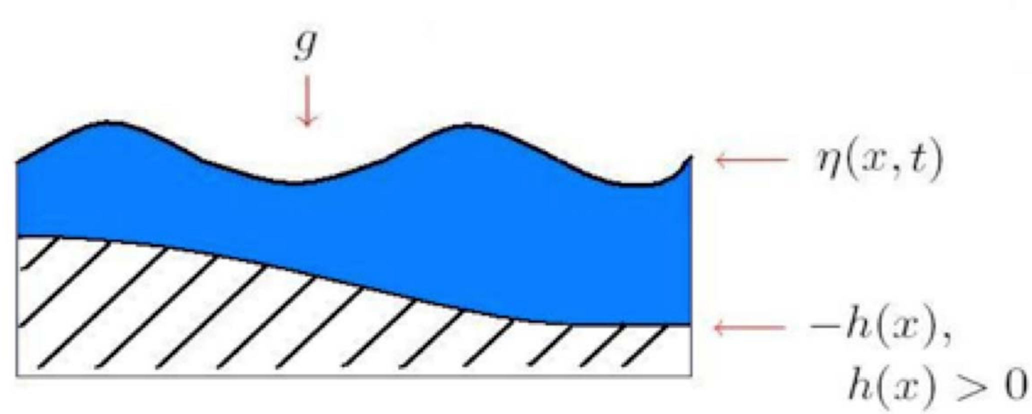
- As the potential temperature refers to the fact that the temperature can be changed with adiabatic expansion or compression, potential vorticity refers to the fact that a parcel's vorticity can be changed by changing latitude (f) or static stability ($-\partial\Theta/\partial p$).

- The expression for PV conservation takes on an even simpler form if we consider a homogeneous, incompressible fluid (ρ constant), such as a shallow tank of water.

- The cross sectional area of the fluid is:

$$A = -g\frac{\delta M}{\delta p} = g\frac{\delta M}{\rho g \delta z} = \frac{\text{constant}}{\delta z}$$

$$\boxed{A = \frac{\text{constant}}{h}}$$



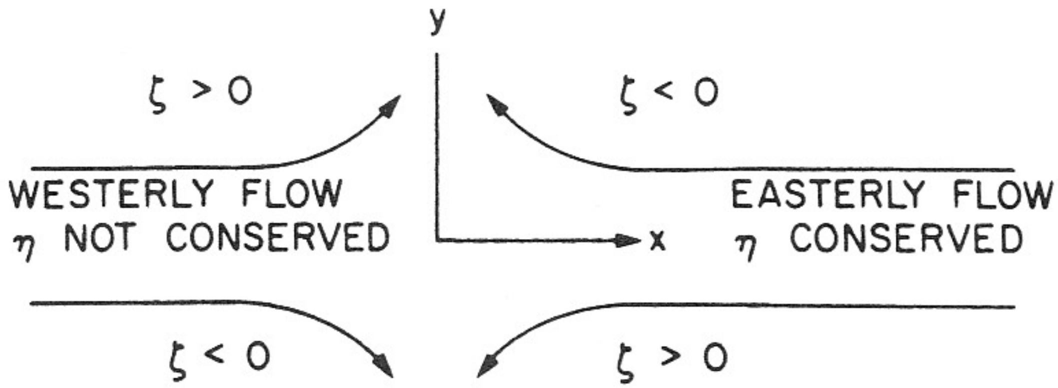
- Substituting the above into $(\zeta_\Theta + f)A = \text{constant}$ gives:

$$\frac{(\zeta + f)}{h} = \frac{\eta}{h} = \text{PV}_{\text{shallow water}} = \text{constant}$$

- In the simplest case where the depth of the fluid is constant, absolute vorticity must be conserved following the motion, which provides a powerful constraint on the flow.

- Suppose we have zonal flow with no relative vorticity such that $\eta = f_0$. Because of the conservation of absolute vorticity, $\eta = \zeta + f = f_0$ at all points along the flow's path.

- If this zonal flow were westerly and curved towards the north (f increasing), we must have $\zeta = f_0 - f < 0$, and if the flow turned to the south (f decreasing), we must have $\zeta = f_0 + f > 0$.



- If this zonal flow were westerly and curved towards the north (f increasing), we must have $\zeta = f_0 - f < 0$, and if the flow turned to the south (f decreasing), we must have $\zeta = f_0 + f > 0$.

- However, as illustrated in the diagram, northward (southward) turning westerly flow implies $\zeta > 0$ (< 0). Thus, westerly flow must remain purely zonal for absolute vorticity to be conserved!

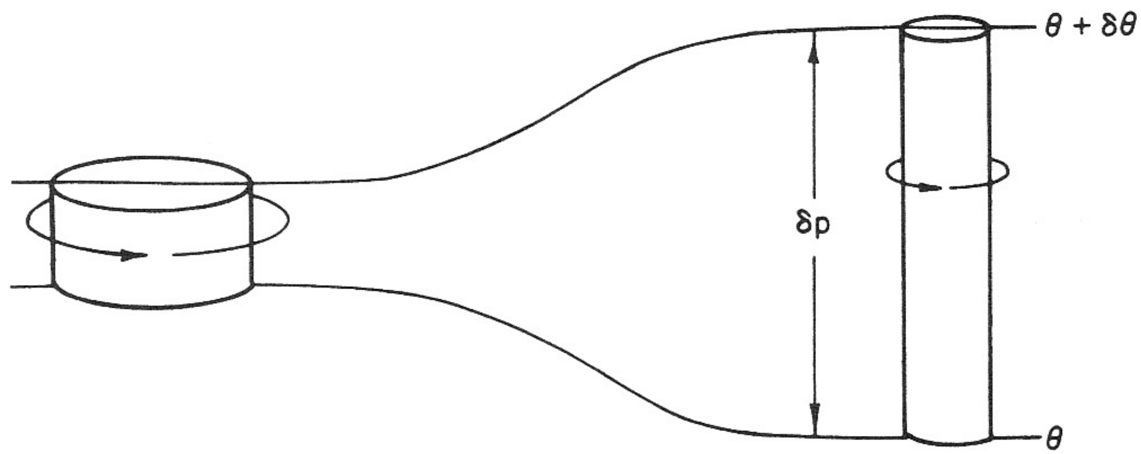
- There is no such constraint on easterly flow, as northward (southward) curvature implies $\zeta < 0$ (> 0) and absolute vorticity is conserved.

- **Returning to the PV conservation equation:**

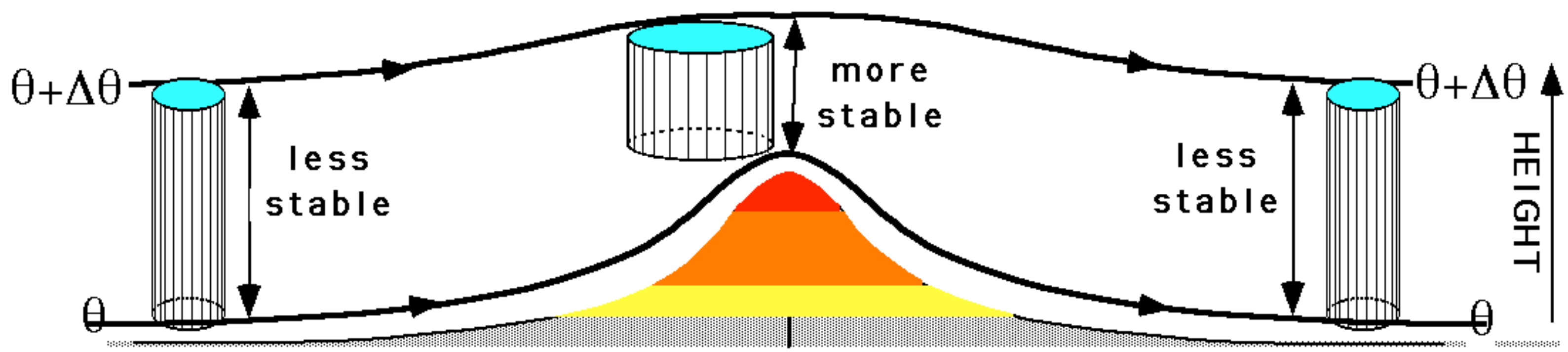
$$\frac{(\zeta + f)}{h} = \frac{\eta}{h} = \mathrm{PV}_{\text{shallow water}} = \text{constant}$$

when the depth of the fluid is not constant, it is potential, rather than absolute, vorticity that is conserved.

- **Considering the equation in this form provides clear physical insight into the nature of PV: If the depth of a rotating column is increased (decreased) by vertical stretching (compression), the absolute vorticity must increase (decrease) so that PV is conserved.**

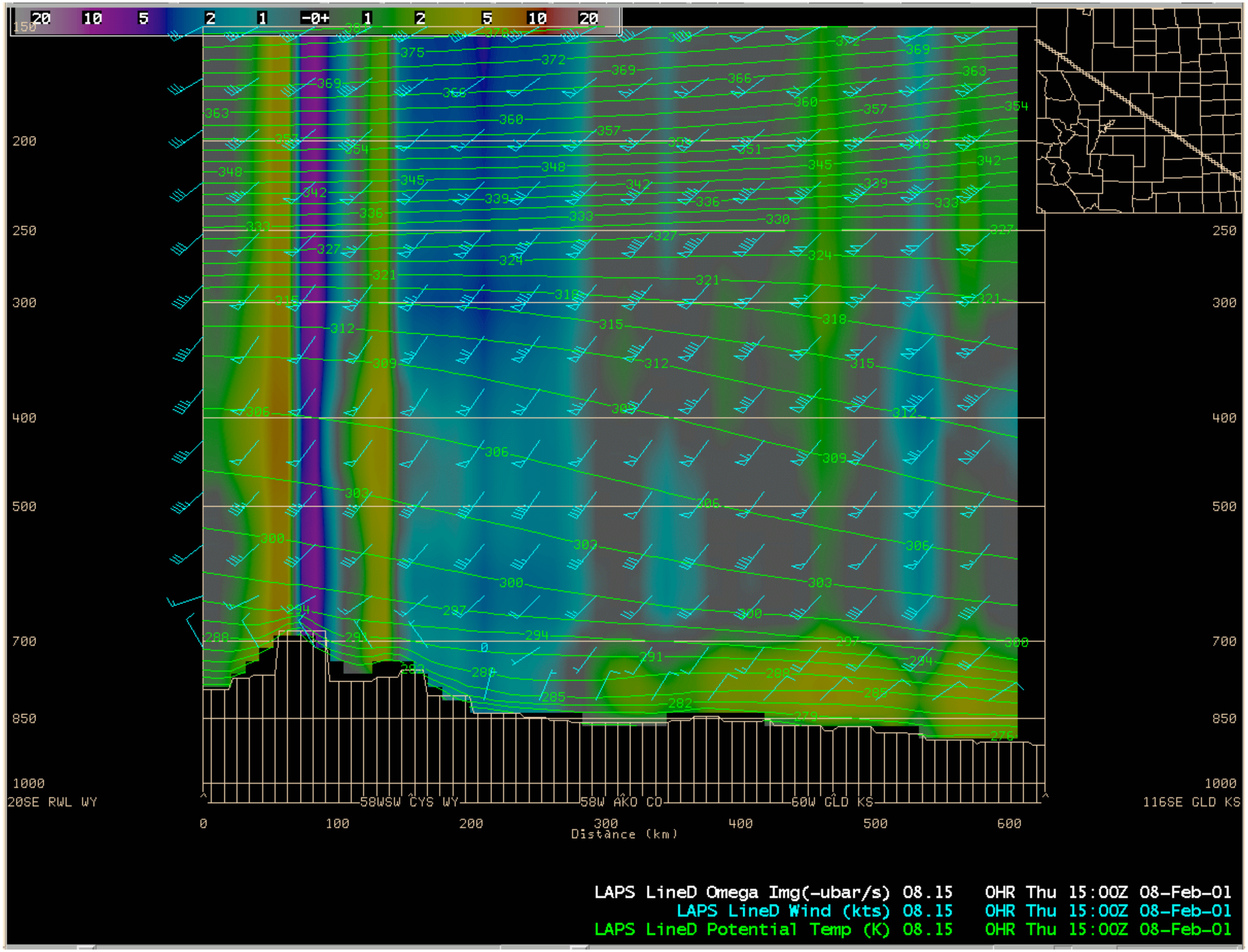


- The classic example of this is flow impinging on an infinitely long mountain chain, where we assume the flow is initially zonal ($\zeta = 0$) and adiabatic such that a vertical column of air of depth h is confined between two isentropes Θ and $\Theta + \Delta\Theta$.



- Because of this last fact, the Θ surface follows the contour of the ground (i.e., it's deflected up and down the mountain) and $\Theta + \Delta\Theta$ is deflected upward as well, but to a lesser extent and the deflection is spread both upstream and downstream from the mountain peak.

- Is this last statement realistic?



20 10 5 2 1 -0+ 1 2 5 10 20
150
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250
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400
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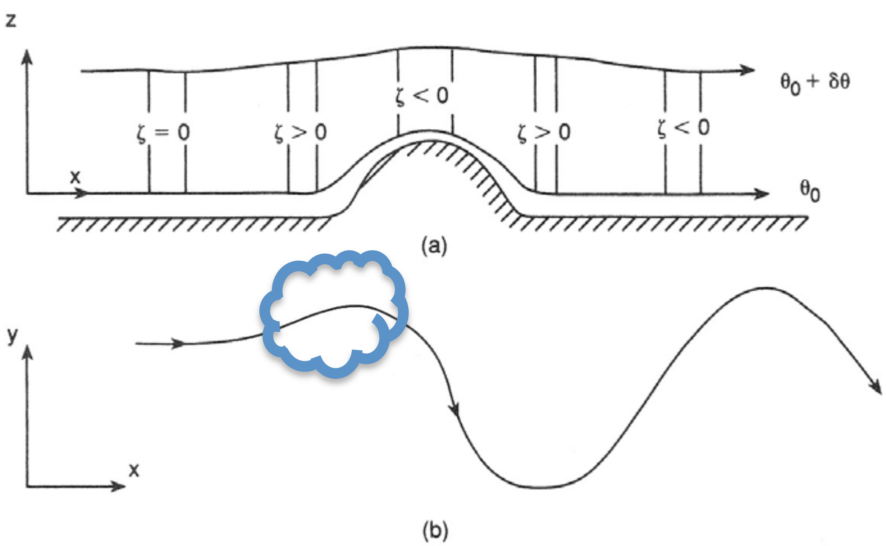
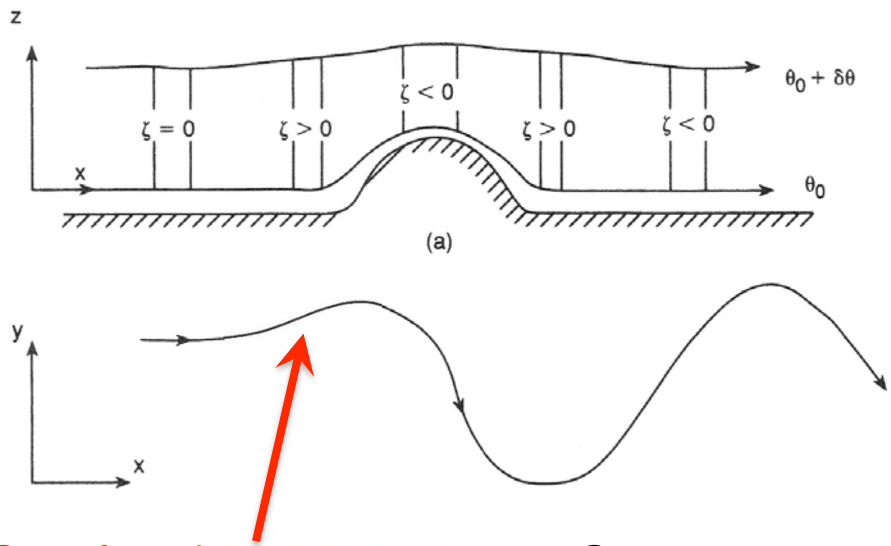
700
850
1000
20SE RWL WY
58WSW CYS WY
58W AKO CO
60W GLD KS
116SE GLD KS
0 100 200 300 400 500 600
Distance (km)
LAPS LineD Omega Img(-ubar/s) 08.15 0HR Thu 15:00Z 08-Feb-01
LAPS LineD Wind (kts) 08.15 0HR Thu 15:00Z 08-Feb-01
LAPS LineD Potential Temp (K) 08.15 0HR Thu 15:00Z 08-Feb-01

- Like for the constant depth (h) case, the evolution of the flow is different for westerly and easterly winds impinging on a mountain.

- Since mid-latitude flow is mostly westerly, we will review the westerly flow case in class and you may study the easterly flow evolution in Holton.



- As indicated to the left, as the flow approaches the barrier, the column is vertically stretched (h increases), ζ must become positive to conserve PV and the flow turns cyclonically as it nears the mountain.



$$\frac{(\zeta + f)}{h} = \frac{\eta}{h} = PV_{\text{shallow water}} = \text{constant}$$

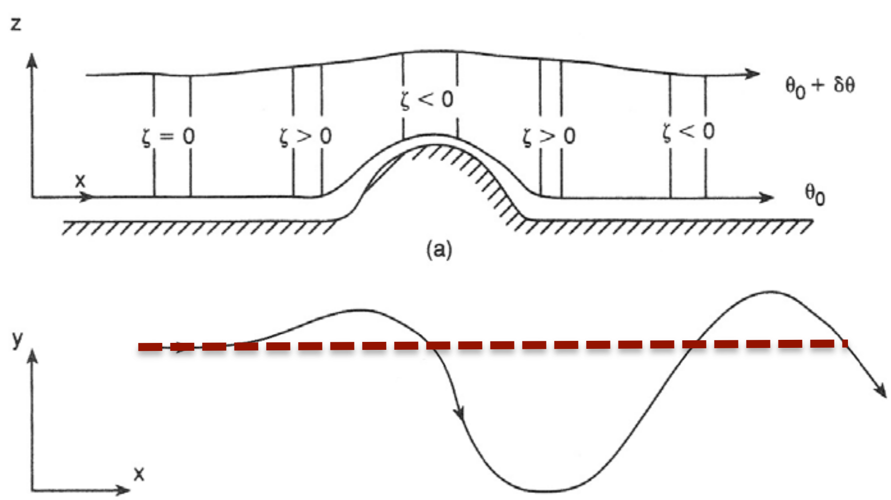
- **Cyclonic curvature for westerly flow = poleward turning, such that f increases as well, reducing the required increase in ζ needed to conserve PV.**

- **As the flow ascends the mountain, h decreases rapidly and ζ must become negative (anticyclonic) indicating southward turning parcels.**

- **As the parcel travels southward and descends the barrier (h increases), it will be at a lower latitude than it's original position and ζ will become large and positive.**

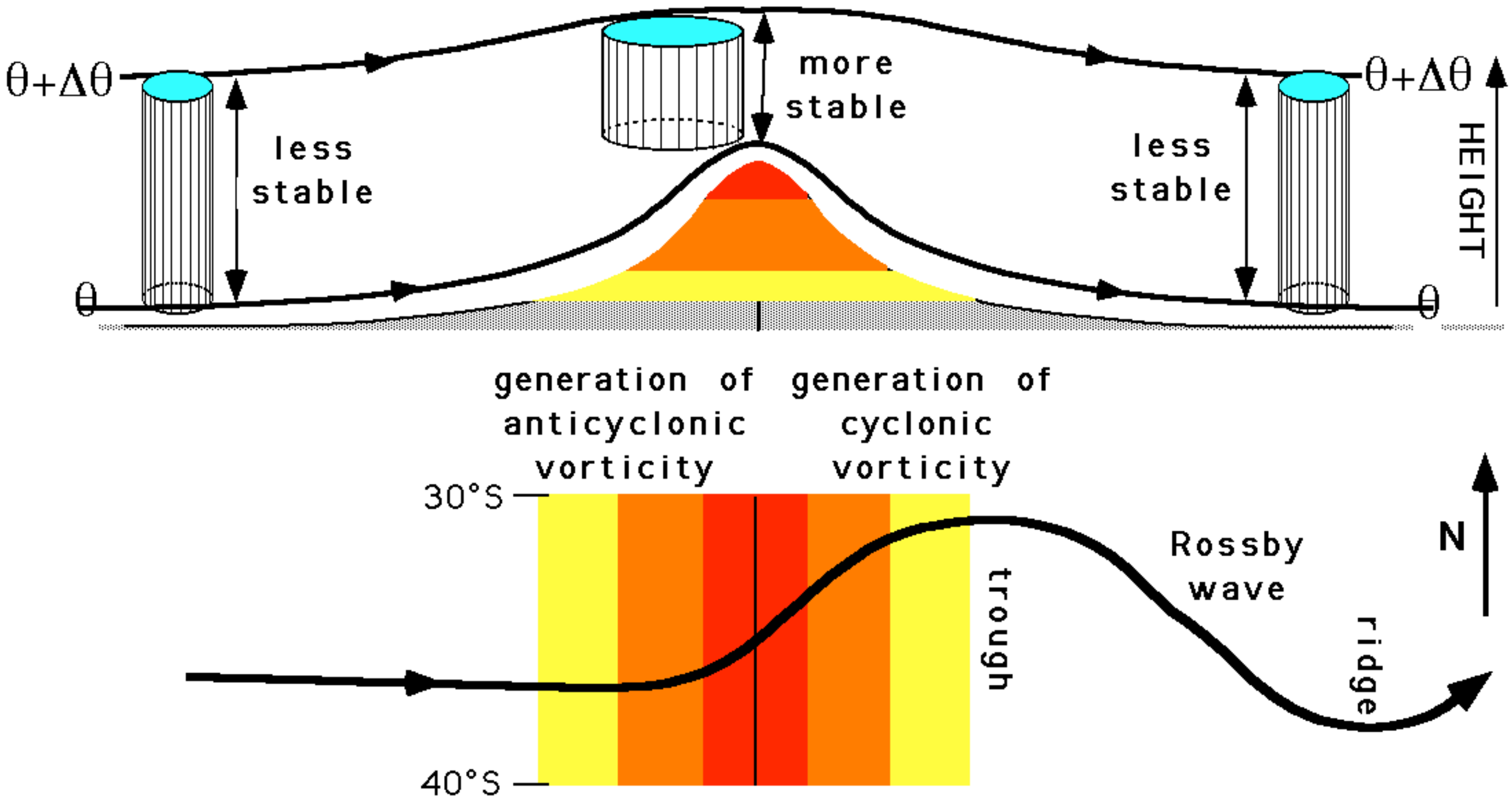
- The trajectory of the parcel thus takes on cyclonic curvature in the lee of the mountain, deflecting to the north.

- Returning to its original latitude, the parcel will still have a poleward component to its motion and will continue poleward with increasing f and decreasing ζ acquiring anticyclonic curvature and reversing its meridional flow direction once again.

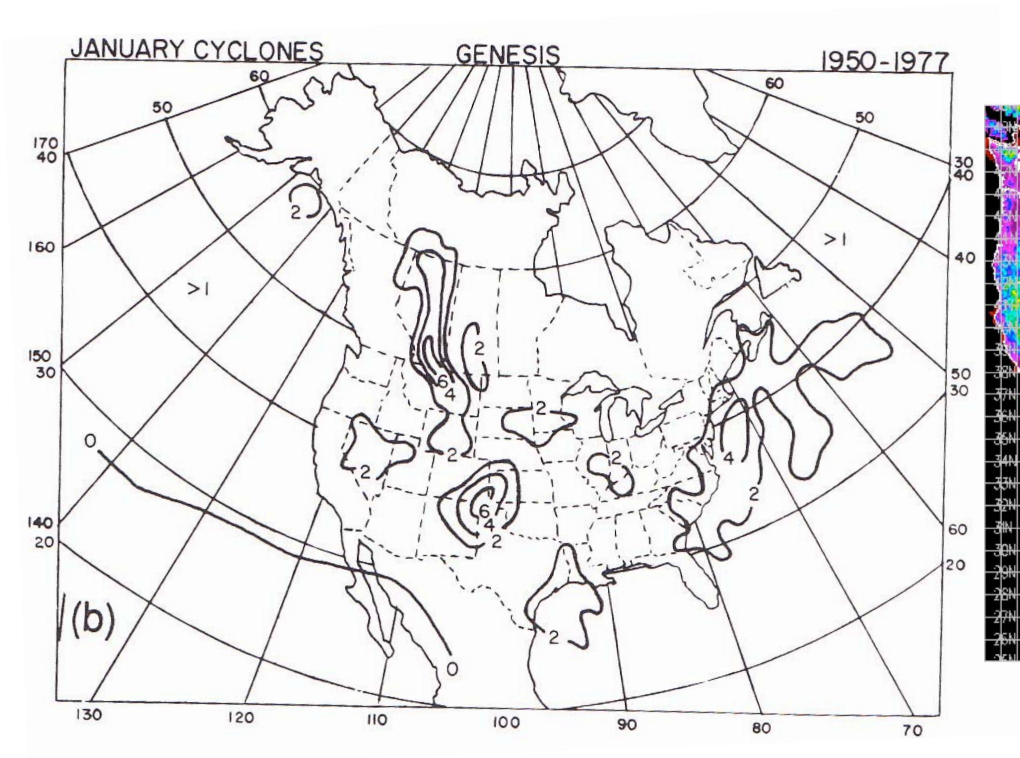


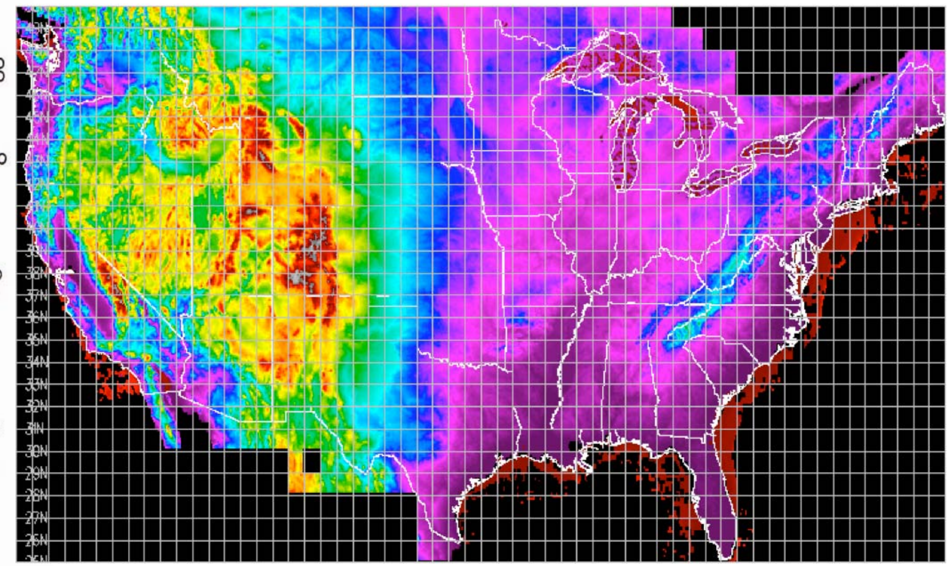
$$\frac{(\zeta + f)}{h} = \frac{\eta}{h} = PV_{\text{shallow water}} = \text{constant}$$

- The depth (h) of the column is now constant, so absolute vorticity (η) must be conserved following the flow, such that a parcel will follow a wave-like trajectory in the horizontal plane downstream of the mountain.



• The meteorological significance of the lee side trough is seen in the figure below detailing the genesis positions of January cyclones (lows, $\zeta > 0$) in the period 1950-1977.





• Clearly, the main cyclogenesis regions are ALL downwind, in the lee of, the major North American mountain ranges, e.g., the Rockies, Appalachians and Sierras.

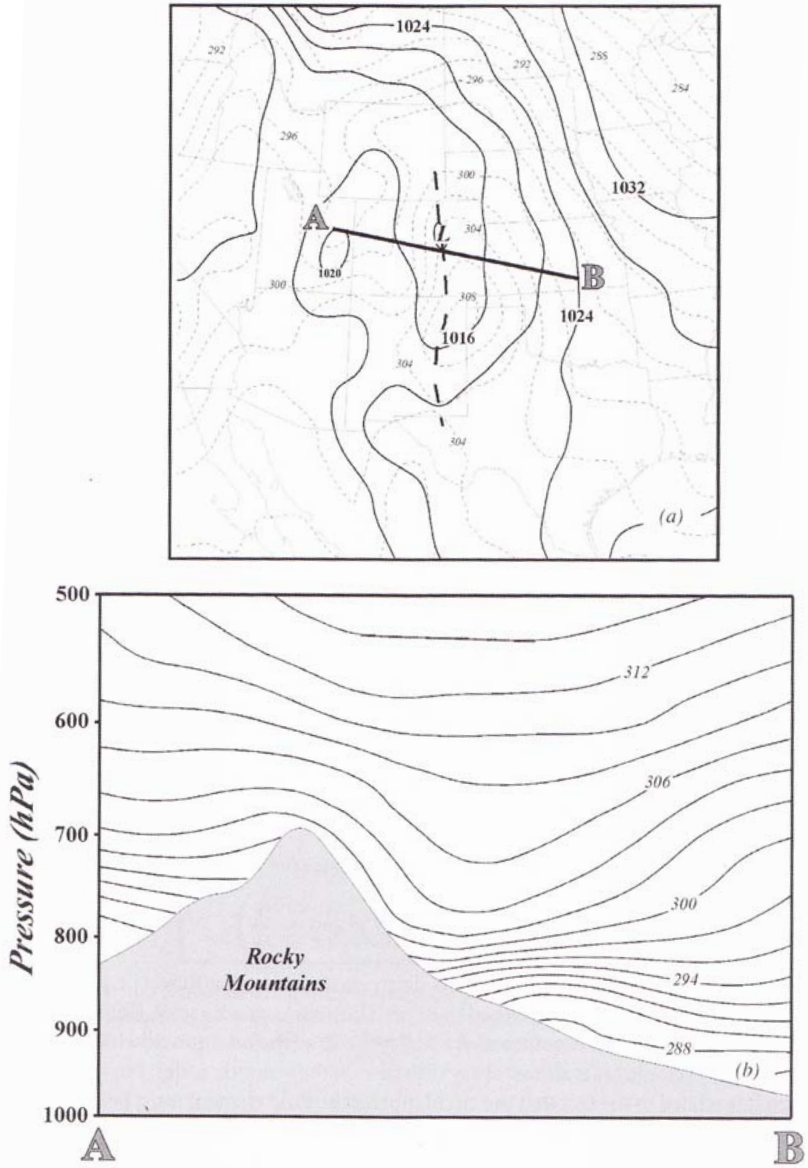


Figure 5.10 (a) Sea-level isobars and 750 hPa potential temperature in the central United States at 0000 UTC 27 December 2004. Thick solid lines are isobars labeled in hPa and contoured every 4 hPa. Dashed lines are isentropes labeled in K and contoured every 2 K. Thick dashed line indicates the axis of the leeside pressure trough. Vertical cross-section along A–B is shown in (b). (b) Vertical cross-section (along line A–B in (a)) of potential temperature. Solid lines are isentropes labeled in K and contoured every 2 K. Note the region of weak stratification in the immediate lee of the Rocky Mountains

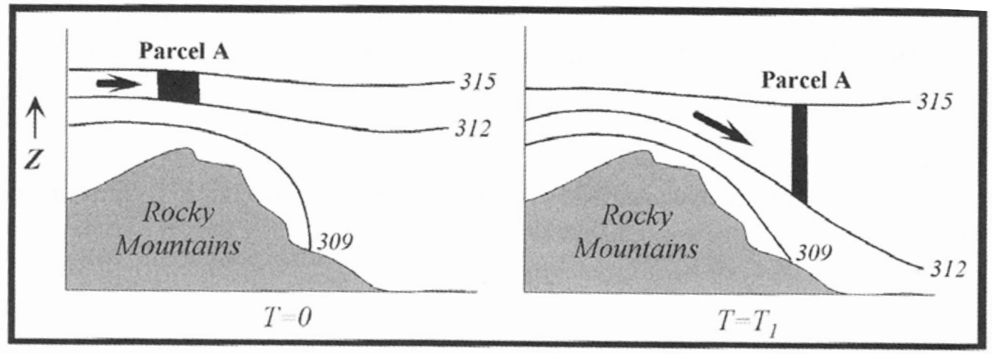


Figure 5.9 Schematic illustration of a fluid column crossing the Rocky Mountains of North America. Parcel A, confined between the 312 and 315 isentropes, is carried across the crest of the mountains by the flow, indicated by the bold arrow, at time $T = 0$. Upon crossing the ridge, the flow subsides and forces a separation between the bounding isentropes ($T = T_1$). Subsidence warming and vorticity increases in the lower troposphere result in the lee of the barrier

- **The lee trough axis also coincides with the axis of warmest air near the surface, a direct result of the adiabatic warming due to subsidence that manifests itself in the vertical separation of the isentropes in the lee.**